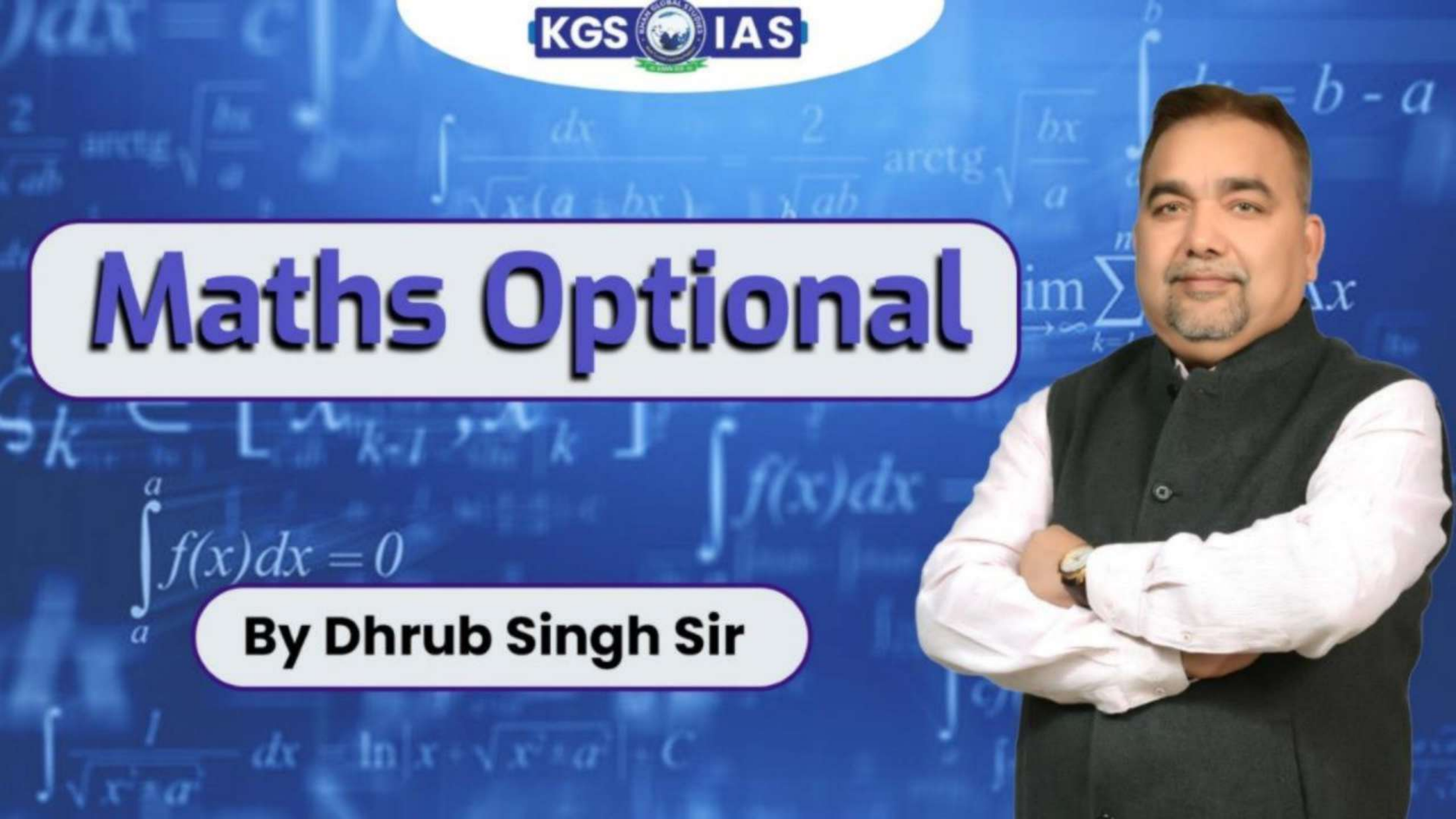
KGS IAS
Maths Optional
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Prob: Find the basis and dimension of the sol^n. space of the following system of linear equations:

$$x+2y-2z+2s-t=0$$
$$x+2y-z+3s-2t=0$$
$$2x+4y-7z+s+t=0$$

$$A=\begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -2 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 3 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim\begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

$r(A) = 2 <$ No. of unknowns (5).

$5 - 2 = 3$ L.I. soln.

writing eqn. in reduced form with the help of echelon form.

$$x + 2y - 2z + 2s - t = 0$$

$$z + s - t = 0$$

$x = k_1,\ s = k_2,\ t = k_3$ $\quad k_1, k_2, k_3$ - scalar

$$z = -k_2 + k_3$$

$$2y = -x + 2z - 2s + t$$

$$= -k_1 - 2k_2 + 2k_3 - 2k_2 + k_3$$

$$2y = -k_1 - 4k_2 + 3k_3$$

$$y = -\frac{1}{2}k_1 - 2k_2 + \frac{3}{2}k_3$$

$= k_1 X_1 + k_2 X_2 + k_3 X_3$

$X_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ \frac{3}{2} \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Sol^n. space

$= L(\{X_1, X_2, X_3\})$

Bases $= \{X_1, X_2, X_3\}$

Dimension $= \underline{3}$.

$$\begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} k_1 \\ -\frac{1}{2}k_1 - 2k_2 + \frac{3}{2}k_3 \\ -k_2 + k_3 \\ k_2 \\ k_3 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 \\ -\frac{1}{2}k_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2k_2 \\ -k_2 \\ k_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{3}{2}k_3 \\ k_3 \\ 0 \\ k_3 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ \frac{3}{2} \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Eigen value and Eigen vector

Let $A = [a_{ij}]_{n \times n}$ be a square matrix.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ be a column vector.

Now consider the matrix eqn.

$$AX = \lambda X \quad \text{—— (I)} \quad \lambda \text{ is a scalar.}$$

$X = 0$ is a soln. of (I)

(I) can be written as:

$$AX = \lambda \cdot I \cdot X$$

$$\Rightarrow AX - \lambda I X = 0$$

$$\Rightarrow (A - \lambda I) X = 0 \quad \text{—— (II)}$$

$$\left.\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \\ a_{n1}x_1 + \cdots\cdots + (a_{nn} - \lambda)x_n &= 0 \end{aligned}\right\} \text{(II)}$$

Roots of $|A-\lambda I|=0$ are called: Ch. roots (Eigen value or Latent roots or proper values). of matrix A.

The set of eigenvalues of A — Spectrum of A

(iii) will have a non-zero soln.

if $r(A-\lambda I)<n$.

i.e. $|A-\lambda I|=0$

Characteristic polynomial

$|A-\lambda I|$ — a polynomial in λ of degree n.

Characteristic eqn: $|A-\lambda I|=0$

Characteristic vector

If λ is a characteristic root of a matrix A of order n then a non-zero vector X s.t

$$AX = \lambda X$$

is called a ch. vector corresponding to λ.

Note: If λ is the eigen value of A then $|A - \lambda I| = 0$

i.e. $A - \lambda I$ is singular.

$\therefore$ $\exists$ a non-zero vector X s.t.

$$(A - \lambda I)X = 0$$

i.e. $AX = \lambda X$

Prob: Determine the characteristic roots of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 + 6\lambda - 4$$

$$|A - \lambda I| = 0$$

$$-\lambda^3 + 6\lambda - 4 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda^2 + 2\lambda - 2) = 0$$

$$\lambda = 2, \quad \lambda^2 + 2\lambda - 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4+8}}{2}$$

$$= \frac{-2 \pm \sqrt{12}}{2} = \frac{-\cancel{2} \pm \cancel{2}\sqrt{3}}{\cancel{2}}$$

$$\lambda = 2, -1 \pm \sqrt{3}$$

Certain relation b/w char. roots and ch. vectors:

Result: λ is a ch. root of a matrix A iff $\exists$ a non-zero vector X s.t. $AX = \lambda X$.

proof: λ is a ch. root of A $\iff$ λ satisfies the eqn. $|A-\lambda I| = 0$

$\iff$ $A - \lambda I$ is singular.

$\iff$ $\exists$ a non-zero vector X s.t.

$$(A-\lambda I)X = 0$$

i.e. $AX - \lambda I X = 0$

i.e. $AX - \lambda X = 0$

i.e. $AX = \lambda X$.

$\Rightarrow K(AX) = K(\lambda X)$

$K \neq 0$ scalar

$\Rightarrow A \cdot (KX) = (K\lambda) X$

$\Rightarrow A \cdot \underline{(KX)} = \lambda \cdot (KX)$

$\therefore$ KX is a ch. vector corr. to the ch. root λ.

Result: If X is a characteristic vector of a matrix A corresponding to the ch. value λ, then KX is also a ch. vector corr. to the same ch. value λ (K is a non-zero scalar).

Proof: $\because$ X is a ch. vector corr. to the ch. root λ.

$\therefore$ $AX = \lambda X$

$\therefore AX = \lambda_1 X$ —— (i)

$AX = \lambda_2 X$ —— (ii) $(X \neq 0)$

(i) & (ii) $\Rightarrow$

$\lambda_1 X = \lambda_2 X$

$\Rightarrow \lambda_1 X - \lambda_2 X = 0$

$\Rightarrow (\lambda_1 - \lambda_2) X = 0$

As $X \neq 0$,

$\lambda_1 - \lambda_2 = 0$

$\Rightarrow \lambda_1 = \lambda_2$.

<u>Note</u>: Corr. to the eigen value λ_1, $\exists$ more than one eigen vector.

<u>Result</u>: If X is a ch. vector of a matrix A, then X can not correspond to more than one ch. value.

<u>Proof</u>: Let X be a ch. vector corr. to the ch. roots λ_1 and λ_2.

Linear independence of ch. vectors corr. to distinct ch. roots

Stat: The ch. vectors corr. to distinct ch. roots of a matrix are linearly independent.

proof: Let $X_1, X_2, \dots X_m$ be the ch. vector corr. to the distinct ch. roots $\lambda_1, \lambda_2, \dots \lambda_m$ of A.

$$\therefore AX_i = \lambda_i X_i \quad \text{—} (1)$$
$$\forall\, i = 1, \dots m.$$

To prove: $X_1, X_2 \dots X_m$ are L.I.

Suppose that $X_1, X_2, \dots X_m$ are L.D.

$\therefore \exists$ some vectors which we suppose as $X_1, X_2, \dots X_r$ $(r < m)$ are L.I.

$\therefore X_1, X_2 \dots X_{r+1}$ are L.D.

(III) $- \lambda_{r+1} \cdot$ (II)

$$k_1(\lambda_1 - \lambda_{r+1})X_1 + k_2(\lambda_2 - \lambda_{r+1})X_2$$

$$+ \cdots + k_r(\lambda_r - \lambda_{r+1})X_r = 0$$

As $X_1, X_2, \ldots X_r$ are L.I

$\therefore k_i(\lambda_i - \lambda_{r+1}) = 0 \quad \forall i = 1, 2, \ldots r$

As λ_i's are distinct.

$\therefore k_i = 0 \quad \forall i = 1, 2, \ldots r$

$\therefore \exists$ scalars $k_1, k_2, \ldots k_{r+1}$ (not all zero) s.t

$$k_1 X_1 + k_2 X_2 + \cdots + k_{r+1} X_{r+1} = 0 \quad \text{— (II)}$$

Pre-multiplying both sides of (II) with A

$$A(k_1 X_1 + k_2 X_2 + \cdots + k_{r+1} X_{r+1}) = A \cdot 0$$

$$\Rightarrow k_1(AX_1) + k_2(AX_2) + \cdots + k_{r+1}(AX_{r+1}) = 0$$

$$\Rightarrow k_1 \lambda_1 X_1 + k_2 \lambda_2 X_2 + \cdots + k_{r+1}\lambda_{r+1} X_{r+1} = 0 \quad \text{— (III)}$$

(using (I))

so (11) $\Rightarrow$

$$k_{r+1} X_{r+1} = 0$$

$$\Rightarrow k_{r+1} = 0 \quad [\text{As } X_{r+1} \neq 0]$$

$$\therefore k_1 = k_2 = \cdots = k_r = k_{r+1} = 0$$

which is a contradiction

$\therefore$ our supposition is wrong.

$\therefore X_1, X_2, \ldots X_m$ are L.I.