KGS IAS

Maths Optional

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Note (1) λ is a ch. root of a non-singular matrix, $\lambda \neq 0$.

(2) At least one ch. root of a singular matrix is '0'.

Result: Show that '0' is a characteristic root of a matrix iff the matrix is singular.

Proof: 0 is a ch. root of a matrix A

iff $\lambda = 0$ satisfies the ch. eqn. $|A - \lambda I| = 0$

i.e. iff $|A| = 0$

i.e. iff A is singular.

Result: The ch. roots of a Hermitian matrix are real.

proof: Let A be a Hermitian matrix.

$$\therefore A^\theta = A \quad \text{--- (1)}$$

Let λ be a ch. root of the matrix A.

$\therefore \exists$ a non-zero vector X s.t.

$$AX = \lambda X \quad \text{--- (2)}$$

Pre-multiplying both sides of (2) with X^θ

$$X^\theta A X = X^\theta(\lambda X) = \lambda X^\theta \cdot X$$

Taking transposed conjugate of both sides,

$$(X^\theta A X)^\theta = (\lambda X^\theta X)^\theta$$

$$\Rightarrow X^\theta A^\theta (X^\theta)^\theta = \bar{\lambda}\, X^\theta \cdot (X^\theta)^\theta$$

$(\lambda A)^{\theta} = \bar{\lambda} \cdot A^{\theta}$

$\Rightarrow x^{\theta} \underline{Ax} = \bar{\lambda} \; x^{\theta} x$ [using ①]

$\Rightarrow x^{\theta} (\lambda x) = \bar{\lambda} \, x^{\theta} x$ [using ②]

$\Rightarrow \lambda \; x^{\theta} x = \bar{\lambda} \, x^{\theta} x$

$\Rightarrow \lambda x^{\theta} x - \bar{\lambda} \, x^{\theta} x = 0$

$\Rightarrow (\lambda - \bar{\lambda}) x^{\theta} x = 0$

[$\because x \neq 0$
$\therefore x^{\theta} \neq 0$
$\therefore x \cdot x^{\theta} \neq 0$]

$\Rightarrow \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda = \bar{\lambda}$

$\Rightarrow \lambda$ is real.

$\therefore$ ch. roots of A are all real.

Result: The ch. roots of a real symmetric matrix are all real.

Proof: Let A be a real symmetric matrix.

$$\therefore A^T = A \quad \text{—— (1)}$$

Now $A^\theta = (\bar{A})^T = A^T$ $\left[\because A \text{ is real} \; \therefore \bar{A} = A\right]$

$= A$ [using (1)]

$\implies$ A is also Hermitian.

Consider iA

$$(iA)^\theta = \bar{i}\, A^\theta$$
$$= -i(-A) \quad \text{(using ①)}$$
$$= iA$$

$\therefore$ iA is a Hermitian matrix.

② $\Rightarrow$

$$i(AX) = i(\lambda X)$$
$$\Rightarrow (iA)X = (i\lambda)X$$

Result: The eigen values of a skew-Hermitian matrix are either purely imaginary or zero.

Proof: Let A be a skew-Hermitian matrix.

$$\therefore A^\theta = -A \quad \text{— ①}$$

Let λ be a ch. root of A.

$\therefore$ $\exists$ a non-zero eigen vector X s.t.

$$AX = \lambda X \quad \text{— ②}$$

$\therefore i\lambda$ is an eigenvalue of iA.

& $\because iA$ is Hermitian.

$\underline{i\lambda}$ is real.

This is possible only when either λ is purely imaginary or zero.

Result: The eigen values of a real skew symmetric matrix are either purely imaginary or zero.

Proof: $\because$ A is skew-symm.

$\therefore A^T = -A$ — (1)

$A^\theta = (\overline{A})^T = (A)^T$ $\left[\because A \text{ is real } \therefore \overline{A} = A\right]$

$= -A$

$\Rightarrow A$ is skew-Hermitian.

We know that ch. roots of a skew Hermitian matrix are either purely imaginary or zero.

$\therefore$ Ch-roots of a real skew-symmetric matrix are also purely imaginary or zero.

Result: The eigen values of a unitary matrix are of unit modulus.

Proof: Let A be a unitary matrix.

$$\therefore \quad A^\theta \cdot A = I \quad \text{—— (i)}$$

Let λ be a ch. root of the matrix A.

$\therefore$ $\exists$ a non-zero vector X

s.t.

$$AX = \lambda X \quad \text{—— (ii)}$$

Taking transposed conjugate of both sides of (ii).

$$(AX)^\theta = (\lambda X)^\theta$$

$$\Rightarrow X^\theta \cdot A^\theta = \bar{\lambda} X^\theta \quad \text{—— (iii)}$$

From (ii) & (iii)

$$(X^\theta A^\theta)(AX) = (\bar{\lambda} X^\theta)(\lambda X)$$

$$\Rightarrow X^\theta (A^\theta A) X = \bar{\lambda}\lambda (X^\theta X)$$

$$\Rightarrow x^{\theta} I x = |\lambda|^2 x^{\theta} x$$

using ①

$$\bar{\lambda} \cdot \lambda = |\lambda|^2$$

$$\Rightarrow x^{\theta} x = |\lambda|^2 x^{\theta} x$$

$$\Rightarrow x^{\theta} x - |\lambda|^2 x^{\theta} x = 0$$

$$\Rightarrow (1 - |\lambda|^2) x^{\theta} x = 0$$

$$\left[\because x \neq 0 \quad \therefore x^{\theta} \neq 0 \quad \therefore x^{\theta} \cdot x \neq 0 \right]$$

$$\Rightarrow 1 - |\lambda|^2 = 0$$

$$\Rightarrow |\lambda|^2 = 1$$

$$\Rightarrow |\lambda| = 1$$

Result: The eigen values of an orthogonal matrix are of unit modulus.

Proof: Real unitary matrices are orthogonal matrices.

$\therefore$ Ch. root of an orthogonal matrix are of unit modulus.

Result: Prove that ± 1 can be the only real eigenvalue of an orthogonal matrix.

Proof: $\because$ Modulus of an eigen value of an orthogonal matrix is 1.

And ± 1 are only real numbers whose modulus is 1.

$\therefore$ ± 1 are only real eigenvalue of an orthogonal matrix.

and so, only possible ch. roots are ± 1.

Result: If A is both real symmetric and orthogonal, prove that all its eigenvalues are $+1$ or -1.

Proof: We know that ch. roots of a real symmetric matrix are real. $\therefore$ ch. roots of A are real.

Also A is orthogonal.

$\therefore$ Modulus of ch. roots of A is 1.

Prob: Find the ch. roots of the 2-rowed orthogonal matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A$$

and verify that they are of unit modulus.

Solⁿ: $|A-\lambda I| = \begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$

$\Rightarrow (\cos\theta-\lambda)^2+\sin^2\theta = 0$

$\lambda^2+\underline{\cos^2\theta}-2\lambda\cos\theta+\underline{\sin^2\theta}=0$

$\Rightarrow \lambda^2-2\lambda\cos\theta+1=0$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta-4}}{2}$$

$$= \frac{\cancel{2}\cos\theta \pm \cancel{2}\sqrt{\cos^2\theta-1}}{\cancel{2}}$$

$= \cos\theta \pm \sqrt{-\sin^2\theta}$

$= \cos\theta \pm i\sin\theta$

$$|\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$|\cos\theta - i\sin\theta| = \sqrt{\cos^2\theta + (-\sin\theta)^2}$$

$$= 1.$$

Prob: Show that the ch. roots of any diagonal matrix are same as its diagonal elements.

Solⁿ: Let $A = \text{Diag}[d_1 \; d_2 \; - - \; d_n]$

$$= \begin{bmatrix} d_1 & 0 & 0 & - - & 0 \\ 0 & d_2 & 0 & - - & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & - - & & d_n \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} d_1-\lambda & 0 & - - & 0 \\ 0 & d_2-\lambda & - - & 0 \\ \vdots & & & \\ - - & & & d_n-\lambda \end{vmatrix} = 0$$

$\Rightarrow (d_1-\lambda)(d_2-\lambda) - - (d_n-\lambda) = 0$

$\Rightarrow \lambda = d_1, d_2, - - d_n.$

$|U-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) \cdots (a_{nn}-\lambda) = 0$$

$$\Rightarrow \underline{\lambda = a_{ii}} \quad \forall i = 1, \dots n.$$

Prob: Ch. roots of a triangular matrix are its diagonal elements.

Sol^n Consider the upper triangular matrix

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Prob: Prove that the square matrices A and A^T have the same eigen value.

Sol^n: $(A-\lambda I)^T = A^T - \lambda \cdot I^T$

$= A^T - \lambda I$ — (1)

Now $|A-\lambda I| = |(A-\lambda I)^T|$

$= |A^T - \lambda I|$

[using (1)]

$|A-\lambda I| = 0$ iff $|A^T - \lambda I| = 0$

$\therefore$ λ is a ch. root of A iff λ is a ch. root of A^T.

$\therefore$ A and A^T have same ch. roots.

if and only if

iff

$$P \iff q$$

$$P \implies q \qquad q \implies p$$