$\bar{A} = [\bar{a}_{ij}]$

Prob: Let A be a square matrix of order 'n' then show that

(i) $|\bar{A}| = \overline{|A|}$ (ii) $|A^{\theta}| = \overline{|A|}$.

Sol^n: Let $A = [a_{ij}]_{n\times n}$ $\bar{A} = [\bar{a}_{ij}]_{n\times n}$

(i) $|\bar{A}| = |\bar{a}_{ij}| = \overline{|a_{ij}|} = \overline{|A|}$

(ii) $|A^{\theta}| = |(\bar{A})^{T}| = |\overline{(A^{T})}| = \overline{|A^{T}|}$

$= \overline{|A|}$

prob: Show that the determinant of a Hermitian matrix is always a real number.

Soln: Suppose A is a Hermitian matrix.

$$\therefore A^\theta = A \quad \text{——— (1)}$$

$$|A| = |A^\theta| = |\overline{(A^T)}| = \overline{|A^T|} = \overline{|A|}$$

$$\Rightarrow |A| = \overline{|A|}$$

$\Rightarrow |A|$ is a real no.

prob: Show that the value of the determinant of a skew symmetric matrix of odd order is always zero.

soln: Suppose A is a skew-symmetric matrix of order n, where n is odd.

$\because A$ is skew-symmetric.

$\therefore A^T = -A$ — (1)

$$|A| = |A^T|$$
$$= |-A|$$
$$= |(-1)A|$$
$$= (-1)^n \cdot |A|$$

$$|KA| = K^n \cdot |A|$$

$[\because n \text{ is odd}]$

$$|A| = -|A|$$
$$\Rightarrow 2|A| = 0$$
$$\Rightarrow |A| = 0$$

$$|AB| = \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 \cdots & & \\ & & \\ & & \end{vmatrix}$$

$$|A|\cdot|B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & \cdots & \\ & & \\ & & \end{vmatrix}$$

Result: If A and B are square matrices, conformable for multiplication then

$$|AB| = |A|\cdot|B|$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \qquad B = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$$

$$AB = \begin{bmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & & \\ & & \\ & & \end{bmatrix}_{3\times 3}$$

Prob: If A and B are square matrices each of order 'n' s.t. $|AB| = 0$ prove that $|A| = 0$ or $|B| = 0$

Sol^n: $\because |AB| = |A| \cdot |B|$

$\because |AB| = 0$

$\Rightarrow |A| \cdot |B| = 0$

$\Rightarrow |A| = 0$ or $|B| = 0$.

$$-\left[a^3+b^3+c^3-3abc\right]$$

~~prove~~ Evaluate

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} \quad R_1 \to R_1+R_2+R_3$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Adjoint of a matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{n \times n}$$

$$\underline{Adj A} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^{T}$$

c_{ij} — cofactor of a_{ij}

Ex 1

$$A = \begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix} \\ \begin{array}{cc} - & + \end{array} \end{array}$$

$$\operatorname{Adj} A = \begin{bmatrix} -4 & -1 \\ 3 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} -4 & 3 \\ -1 & 2 \end{bmatrix}$$

Result: If A be any n-rowed square matrix then $A\cdot(\operatorname{Adj} A) = (\operatorname{Adj} A)\cdot A = |A|\cdot I_n$

proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & -- & a_{nn} \end{bmatrix}_{n\times n}$

$$\operatorname{Adj} A = \begin{bmatrix} A_{11} & A_{12} & -- & A_{1n} \\ A_{21} & A_{22} & -- & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & -- & A_{nn} \end{bmatrix}^T \qquad A_{ij} \text{ — cofactor of } a_{ij}$$

$$= \begin{bmatrix} A_{11} & A_{21} & -- & A_{n1} \\ A_{12} & A_{22} & -- & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & -- & A_{nn} \end{bmatrix}$$

$$A\cdot(\operatorname{Adj} A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$(i,j)^{th}$ element of

$$A\cdot(\operatorname{Adj} A) = a_{i1}\cdot A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}$$

$$= \begin{cases} |A|, & i=j \\ 0, & i\neq j \end{cases}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\therefore A \cdot (\operatorname{Adj} A) = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & |A| \end{bmatrix}$$

$$= (|A|) \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\underline{A \cdot (\operatorname{Adj} A) = |A| \cdot I_n}$$

Similarly, we can prove that

$$(\operatorname{Adj} A) \cdot A = |A| \cdot I_n$$

$$\therefore A (\operatorname{Adj} A) = (\operatorname{Adj} A) \cdot A = |A| \cdot I_n$$

<u>Note</u>: If $|A| \neq 0$

then

$$\frac{1}{|A|} (A \cdot \operatorname{Adj} A) = \frac{1}{|A|} \cdot$$

$$(\operatorname{Adj} A \cdot A) = I_n$$

Inverse of a matrix: A matrix B is called the inverse of A if

$$AB = BA = I.$$

B is called inverse of A, denoted by A^{-1}.

Note: (i) Every square matrix need not have inverse.

(ii) A — invertible matrix

A^{-1} exists.

Note (i) $I^{-1} = I$.

$\therefore I \cdot I = I \cdot I = I$

(ii) $(A^{-1})^{-1} = A$

As $A \cdot A^{-1} = A^{-1} \cdot A = I$

Let $B = A^{-1}$

$\therefore AB = BA = I$

$\therefore B^{-1} = A$

$(A^{-1})^{-1} = A$.

Result: Every invertible matrix has unique inverse.

proof: Let A be an invertible matrix.

Let B and C are inverses of A.

$\therefore AB = BA = I$

$AC = CA = I$

Now $B = B \cdot I = B(AC) = (BA) \cdot C$

$= I \cdot C$

$= C$.

So Inverse is unique.

iffs

$P \Longleftrightarrow Q$

P true $\Longrightarrow$ Q true

Q true $\Longrightarrow$ P true.

Result: The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

Proof: Suppose A has inverse.

$$\therefore \quad A \cdot A^{-1} = A^{-1} \cdot A = I.$$

$$A \cdot A^{-1} = I$$

$$\Longrightarrow \quad |A \cdot A^{-1}| = |I|$$

$$\Longrightarrow \quad |A| \cdot |A^{-1}| = 1 \quad \Longrightarrow |A| \neq 0$$

Let $B = \dfrac{\operatorname{Adj} A}{|A|}$.

$① \Longrightarrow$

$$AB = BA = I$$

$$\therefore A^{-1} = B = \frac{\operatorname{Adj} A}{|A|}$$

$$\boxed{= \frac{1}{|A|}(\operatorname{Adj} A)}$$

$\therefore$ A has inverse.

<u>Conversely</u> suppose that $|A| \neq 0$.

we have

$$A \cdot \operatorname{Adj} A = (\operatorname{Adj} A) \cdot A = |A| \cdot I$$

$$\Rightarrow \frac{1}{|A|}\left(A \cdot \operatorname{Adj} A\right) = \frac{1}{|A|}\left[(\operatorname{Adj} A) \cdot A\right] = I$$

$$\Rightarrow A \cdot \left(\frac{\operatorname{Adj} A}{|A}\right) = \left(\frac{\operatorname{Adj} A}{|A|}\right) \cdot A = I \text{ --- } ①$$

Non-singular Matrix: A square matrix A is called non-singular if $|A| \neq 0$.

Note: (i) A — singular matrix.

$|A| = 0$

(ii) A non-singular matrix has inverse.

(iii) A matrix is invertible if it is non-singular.

prob: Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$

$$\longrightarrow \quad A^{-1} = \frac{1}{|A|} \times \text{Adj } A$$

$$A^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Result: If A, B be two n-rowed non-singular matrix then AB is non-singular and $(AB)^{-1} = B^{-1} \cdot A^{-1}$.

Proof: $\because$ A and B are non-singular

$\therefore$ $|A| \neq 0 \quad |B| \neq 0$

$$|AB| = |A|\,|B|$$

$$\neq 0$$

$\therefore$ AB is non-singular.

Let $C = B^{-1} A^{-1}$.

$$\begin{aligned}
(AB)C &= A\cdot[BC] \\
&= A[B(B^{-1}A^{-1})] \\
&= A[(BB^{-1})\cdot A^{-1}] \\
&= A[I\cdot A^{-1}] \\
&= A\cdot A^{-1} = I
\end{aligned}$$

Similarly, $C(AB) = I$

$$\boxed{\begin{aligned}(AB)^{-1} &= C \\ &= B^{-1}A^{-1}.\end{aligned}}$$

Let $B = \underline{(A^{-1})^T}$

① $\Rightarrow$

$B \cdot A^T = A^T \cdot B = I$

$$\boxed{\begin{aligned}(A^T)^{-1} &= B \\ &= (A^{-1})^T\end{aligned}}$$

Result: If A be an $n \times n$ non-singular matrix then $(A^T)^{-1} = (A^{-1})^T$.

$\rightarrow$ $\because$ A is non-singular.

$\therefore$ $|A| \neq 0$

And A^{-1} exists.

$A A^{-1} = A^{-1} \cdot A = I$

$\Rightarrow (A \cdot A^{-1})^T = (A^{-1} \cdot A)^T = I$

$\Rightarrow (A^{-1})^T \cdot A^T = A^T \cdot (A^{-1})^T = I$ — ②

Prob: If A be an $n \times n$ non-singular matrix then $(A^{-1})^{\theta} = (A^{\theta})^{-1}$.