KGS IAS

Maths Optional

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$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 3C_1$: $\boxed{E'_{21}(-3)}$

$C_3 \to C_3 - 3C_1$: $\boxed{E'_{31}(-3)}$

$A \sim I_3$

$\rho(A) = \rho(I_3) = 3$

$$E_{31}(-1) \cdot E_{21}(-1) \cdot A \, E'_{21}(-3) \cdot E'_{31}(-3) = I_3.$$

Prob: Express $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ as a product of elementary matrices.

Sol^n.

Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$: $\boxed{E_{21}(-1)}$

$R_3 \to R_3 - R_1$: $\boxed{E_{31}(-1)}$

$$\{E_{ij}(k)\}^{-1} = E_{ij}(-k)$$

$$E_{31}(-1)\cdot E_{21}(-1)\, A \cdot E'_{21}(-3) \cdot E'_{31}(-3) = I_3$$

$$\Rightarrow A = E_{21}(-1)^{-1}\, E_{31}(-1)^{-1}\, I_3 \{E'_{31}(-3)\}^{-1} \{E'_{21}(-3)\}^{-1}$$

$$= E_{21}(1) \cdot \underline{E_{31}(1)} \cdot \underline{I_3}\, E'_{31}(3)\, E'_{21}(3)$$

$$A = E_{21}(1) \cdot E_{31}(1) \cdot E'_{31}(3)\, E'_{21}(3)$$

(I) Reflexive

$A \sim A \quad \forall A \in M$

(II) Symmetric

Let $A \sim B$

$\Rightarrow B \sim A$

(III) Transitive

$A \sim B$ & $B \sim C$

$\Rightarrow A \sim C$

Equivalence of matrices: Let A and B are two $m \times n$ matrices. If B is obtained from A by a finite number of E-transformation of A, then we say that A is equivalent to B.

Notation: $A \sim B$

Note: The relation '$\sim$' in the set M of all $m \times n$ matrices is an equivalence set!

Note ① If $A \sim B$ then

$r(A) = r(B)$.

② If A and B are equivalent matrices then $\exists$ non-singular matrices P & Q s.t.

$$PAQ = B.$$

Row equivalence

$$A \xrightarrow{\text{Through only E-row trans.}} B$$

$$A \overset{R}{\sim} B$$

Column equivalence

$$A \xrightarrow{\text{Through only E-col. trans.}} B$$

$$A \overset{C}{\sim} B$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad R_3 \to R_3 + \frac{3}{2} R_2$$

In Echelon form.

$\therefore \, r(A) = 3 = r(I_3)$

$\therefore \, A \sim I_3$.

Note 3: If A and B are of same order and $r(A) = r(B)$ then $A \sim B$.

prob: Is the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ equivalent to I_3?

Solⁿ

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -3 & -5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

Prob: Find the rank of the matrix

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

by using E-row trans.

Prob: Is the pair of matrices

$$\begin{bmatrix} 4 & 0 & 2 \\ 3 & 1 & -2 \\ -5 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 3 & 9 & 0 & 2 \\ 7 & -2 & 0 & 1 \\ 8 & 7 & 4 & 5 \end{bmatrix}$$

equivalent?

Ans: No, Matrices are not of same order.

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & -6 & 1 \end{bmatrix} \quad R_3 \to R_3 - \frac{3}{2} R_1$$

$$\sim \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 10 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

In echelon form

$\rho(A) = 3.$

we know that every non-singular matrix can be written as a product of finite number of E-matrices.

let $Q = Q_1 Q_2 \cdots Q_t$

where Q_i is a non-singular matrix $\forall i = 1 \cdots t$.

Th: If A be an $m \times n$ matrix of rank 'r' then there exists a non-singular matrix P such that $PA = \begin{bmatrix} G \\ O \end{bmatrix}$, where G is an $r \times n$ matrix of rank 'r' and O is $(m-r) \times n$ zero matrix.

proof: Given that $r(A) = r$

$\therefore \exists$ non-singular matrices P & Q

s.t.

$$PAQ = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right] \quad \text{——— (1)}$$

we have

$$PA = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right] Q_t^{-1} Q_{t-1}^{-1} \cdots Q_1^{-1}$$

$$\Rightarrow PA = \begin{bmatrix} G \\ O \end{bmatrix} \quad \text{--- (iii)}$$

where G is $r \times n$ matrix and O is $(m-r) \times n$ matrix.

$\therefore$ (I) $\Rightarrow$

$$PAQ_1Q_2 \cdots Q_t = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right] \quad \text{--- (ii)}$$

$\because$ Each column trans. is equivalent to post-multiplying with the corresponding E-matrix.

Clearly no column trans. can change the last $(m-r)$ rows of $\left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right]$.

Now post-multiplying both sides of (ii) successively by $Q_t^{-1}, Q_{t-1}^{-1}, \cdots Q_1^{-1}$,

Proof: A is an $m\times n$ matrix of rank 'r'.

$\therefore \exists$ non-singular matrices P and Q s.t

$$PAQ = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right] \quad -(1)$$

$\because$ Each non-singular matrix is a product of E-matrices.

Let $P = P_s \cdot P_{s-1} \cdots P_1$

$\because$ Elementary trans. do not change the rank of a matrix.

$\therefore r(G) = r(I_r) = r$.

Th: If A be an $m\times n$ matrix of rank 'r' then there exists a non-singular matrix Q s.t.

$AQ = [H \;\; O]$, where H is an $m\times r$ matrix of rank 'r' and O is a zero matrix of order $m\times(n-r)$.

we set

$$A\,Q = P_1^{-1} \cdots P_{s-1}^{-1} P_s^{-1} \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right] \quad \text{(III)}$$

$\because$ Each elementary R-trans. will not change the last $(n-r)$ columns of $\left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right]$.

where $P_1, P_2, \cdots P_s$ are elementary matrices.

$\therefore$ (I) $\Rightarrow$

$$P_s P_{s-1} \cdots P_1 A\,Q = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right] \quad \text{(II)}$$

$\because$ Each row-transf. is equivalent to pre-multiplying with corresponding E-matrix.

Pre-multiplying both sides of (II) successively with $P_s^{-1}, P_{s-1}^{-1}, \cdots P_1^{-1}$,

$\therefore$ (iii)

$$A\phi = [H \quad O],$$

$H_{m \times r}$ — matrix

$O_{m \times (n-r)}$ — Matrix

$\because$ Elementary trans. do not change the rank of a matrix.

$$r(H) = r(I_r) = r.$$

Thus $\exists$ a non-singular matrix ϕ s.t. $A\phi = [H \quad O]$.