KGS IAS

Maths Optional

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$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$r(A) = 2$

$r([A:B]) = 3$

$\because r(A) \neq r([A:B])$

No sol^n.

Inconsistent

Prob: Investigate for what values of λ and μ the equations

$$x+y+z=6$$

$$x+2y+3z=10$$

$$x+2y+\lambda z=\mu$$

have (i) No sol^n. (ii) a unique sol^n (iii) infinitely many sol^n.

$$[A:B]=\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$r(A) = r([A:B])$
$= 2 <$ No. of unknowns.

Infinitely many soln.

(I) $\lambda = 3, \mu \neq 10$

$r(A) = 2, \quad r[A:B] = 3$

No. soln.

(II) $\lambda \neq 3, \quad \mu$ — arbitrary.

$r(A) = r([A:B]) = 3 =$ No. of unknowns

unique soln.

(III) $\lambda = 3, \quad \mu = 10$

Prob: For what values of λ the equations

$$x+y+z=1$$
$$x+2y+4z=\lambda$$
$$x+4y+10z=\lambda^2$$

have a soln. and solve them completely in each case.

$$[A:B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

Case (1) $\lambda = 1$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

writing eqn. using echelon form,

$$x + y + z = 1$$
$$y + 3z = 0$$

let $z = k$, $k \in \mathbb{R}$

$$y = -3k$$

$$x = 1 + 3k - k$$
$$= 1 + 2k$$

$r(A) = 2$

$r(A:B) = 2$ only if

$$\lambda^2 - 3\lambda + 2 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0$$
$$\Rightarrow \lambda(\lambda - 2) - 1(\lambda - 2) = 0$$
$$\Rightarrow (\lambda - 1)(\lambda - 2) = 0$$
$$\Rightarrow \lambda = 1, 2$$

$\boxed{\lambda = 2}$

$z = c$, $y = 1 - 3c$
$x = 2c$
$c \in \mathbb{R}$

Cramer's rule

3 non-homogeneous eqn. in three unknowns.

$$a_{11}x + a_{12}y + a_{13}z = b_1 \quad \text{---} \quad (\mathrm{I})$$

$$a_{21}x + a_{22}y + a_{23}z = b_2 \quad \text{---} \quad (\mathrm{II})$$

$$a_{31}x + a_{32}y + a_{33}z = b_3 \quad \text{---} \quad (\mathrm{III})$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

Let

Cofactor of $a_{11} = A_{11}$

" " $a_{21} = A_{21}$

" " $a_{3} = A_{31}$

$$(\mathrm{I}) \times A_{11} + (\mathrm{II}) \times A_{21} + (\mathrm{III}) \times A_{31}$$

$$(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})x$$

$$+ (a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31})y$$

$$+ (\qquad)z = b_1A_{11} + b_2A_{21} + b_3A_{31}$$

$$\Delta x + 0 \cdot y + 0 \cdot z = \Delta_1$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$$\Delta \cdot x = \Delta_1$$

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$x = \frac{\Delta_1}{\Delta}$$

Similarly, $y = \dfrac{\Delta_2}{\Delta}$, $z = \dfrac{\Delta_3}{\Delta}$

Prob: using Cramer's rule

Solve

$$x+y+z=6$$

$$x-y+z=2$$

$$2x-y+3z=9$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & -3 & 1 \end{vmatrix} \quad \begin{array}{l} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{array}$$

$$= 1(-2-0)$$

$$= -2$$

$$\Delta_1 (\Delta_x) = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 9 & -1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 1 & 1 \\ 8 & 0 & 2 \\ 15 & 0 & 4 \end{vmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$= -1(32-30)$$

$$= -2$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 9 & 3 \end{vmatrix} = -4$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 9 \end{vmatrix} = -6$$

$$\boxed{x = \frac{\Delta_1}{\Delta} = \frac{-2}{-2} = 1 \quad y = \frac{\Delta_2}{\Delta} = \frac{-4}{-2} = 2 \quad z = 3}$$

prob: Solve the equations

$$\lambda x + 2y - 2z = 1$$

$$4x + 2\lambda y - z = 2$$

$$6x + 6y + \lambda z = 3.$$

Also discuss the case when $\lambda = 2$.

HW

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note: (1) Homo. syst. Always consistent.

(11) $x_1 = x_2 = \cdots = x_n = 0$ trivial soln.

Homogenous system of Linear eqns:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} \text{—— (1)}$$

(1) can be expressed as.

$$AX = O \quad \text{—— (2)}$$

proof: $\because X_1$ & X_2 are sol^n. of $AX=0$

$\therefore AX_1=0,\ AX_2=0$ — (1)

$A(k_1X_1+k_2X_2)$

$=k_1(AX_1)+k_2(AX_2)$

$=k_1\cdot 0+k_2\cdot 0$

$=0$

$\therefore k_1X_1+k_2X_2$ is a sol^n of $AX=0$

(iii) Any other sol^n. (other than all zeros) — Non-trivial sol^n.

Result: If X_1 and X_2 are two sol^n. of $AX=0$ then their linear combination $k_1X_1+k_2X_2$ (k_1, k_2 arbitrary number) is also a sol^n. of $AX=0$.

Note: The collection of all soln. of $AX=0$ forms a sub-space of the vector space $V_n(F)$ (or $F^{(n)}$)

Theorem: The number of linearly independent soln. of the linear system $AX=0$ is $n-r$, where r is the rank of the matrix A.

$$\alpha \begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \alpha \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\alpha = 0$