Trace of a matrix:

$A = [a_{ij}]_{n \times n}$ — a square matrix.

(i) $tr(KA) = K \cdot tr(A)$

(ii) $tr(A+B) = tr(A) + tr(B)$

(iii) $tr(AB) = tr(B \cdot A)$

$$tr(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}.$$

$$= \sum_{i=1}^{n} a_{ii}.$$

Ex (1)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

$$tr(A) = 1 + (-1) + 4 = \underline{4}.$$

$$A = [a_{ij}]_{n\times n} \qquad B = [b_{ij}]_{n\times n}$$

$$A + B = [a_{ij} + b_{ij}]_{n\times n}$$

$$tr(A+B) = \sum_{i=1}^{n} (a_{ii} + b_{ii})$$

$$= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$

$$= \underline{tr(A) + tr(B)}$$

$A = [a_{ij}]_{m \times n}$

Transpose of a matrix.

$A = [a_{ij}]_{m \times n}$

$B = A^T$

$B = [b_{ij}]_{n \times m}$

$b_{ij} = a_{ji}$

$\forall i, j$

A' or A^T — $n \times m$

$(i, j)^{th}$ element of $A^T = (j, i)^{th}$ element of A

Ex:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 0 \end{bmatrix}_{3 \times 2}$$

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}_{2 \times 3}$$

I. $(A^T)^T = A$

II. $(A \pm B)^T = A^T \pm B^T$.

III. $(KA)^T = K \cdot A^T$

IV. $(AB)^T = B^T \cdot A^T$.

Symmetric Matrix.

$A = [a_{ij}]_{n \times n}$

A square matrix A is called a symmetric matrix if $a_{ij} = a_{ji} \ \forall i, j$.

or

$A = A^T$.

Ex: (1) $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = A$

A – symmetric matrix.

(11) $B = \begin{bmatrix} a & f & g \\ f & b & h \\ g & h & c \end{bmatrix}$

$B^T = \begin{bmatrix} a & f & g \\ f & b & h \\ g & h & c \end{bmatrix}$

$B^T = B$

$\therefore$ B is a sym. matrix

Skew-symmetric matrix:

A square matrix $A = [a_{ij}]_{n \times n}$ is called a skew-symmetric matrix if

$$A^T = -A.$$

i.e. $a_{ij} = -a_{ji} \quad \forall i, j.$

$a_{ii} = -a_{ii} \quad \forall i$

$\Rightarrow 2a_{ii} = 0 \quad \forall i$

$\Rightarrow \underline{a_{ii} = 0}, \quad \forall i$

Diagonal elements must be zero.

<u>Ex:</u> $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -A$

Prob: If A is a symmetric matrix, then KA is also symmetric. Prove.

Sol^n: $\because$ A is a symmetric matrix.

$\therefore$ $A^T = A$

$$(KA)^T = K \cdot A^T$$
$$= K \cdot A$$

$\therefore$ KA is a symmetric matrix.

<u>Prob</u>: If A and B are symmetric matrices then prove that A+B is also a symmetric matrix.

<u>Solⁿ</u>: $\because$ A and B are symmetric matrices.

$$\therefore A^T = A, \quad B^T = B \longrightarrow (1)$$

$$(A+B)^T = A^T + B^T$$

$$= A + B \quad [\text{using } (1)]$$

$\therefore$ A+B is a symm. matrix

Prob: If A and B are skew-symmetric matrices then prove that A+B is also a skew-symmetric matrix.

Soln: $\because$ A and B — skew-symmetric.

$\therefore A^T = -A, \; B^T = -B$ —— (1)

$$(A+B)^T = A^T + B^T$$
$$= -A - B \quad [\text{by (1)}]$$
$$= -(A+B)$$

$\therefore$ A+B — skew-symmetric.

$$(AB-BA)^T$$
$$=(AB)^T-(BA)^T$$
$$=B^T\cdot A^T-A^T\cdot B^T$$
$$=BA-AB$$
$$=-(AB-BA)$$

$\therefore AB-BA$ skew-symmetric

prob: If A and B are symmetric matrices then prove that $AB+BA$ is symmetric and $AB-BA$ is skew-symmetric.

$\because$ A and B — symmetric.

$\therefore A^T=A,\ B^T=B \longrightarrow (1)$

$$(AB+BA)^T=(AB)^T+(BA)^T$$
$$=B^TA^T+A^T\cdot B^T$$
$$=BA+AB \quad (\text{using } (1))$$
$$=AB+BA$$

Prob: If A and B are symmetric matrices then AB is symmetric iff A and B commute i.e. $AB = BA$.

Sol^n: $\because$ A and B are symmetric

$\therefore A^T = A, \quad B^T = B \longrightarrow (1)$

Suppose AB is symmetric

$$AB = (AB)^T = B^T \cdot A^T$$
$$= BA \quad [\text{using } (1)]$$

Conversely

Suppose $AB = BA$

$$(AB)^T = B^T A^T$$
$$= BA \quad [\text{using } (1)]$$
$$= AB$$

$\therefore AB$ is symmetric.

Prob: If A be any matrix then prove that $A \cdot A^T$ and $A^T \cdot A$ are symmetric matrices.

Sol^n:

$$(A \cdot A^T)^T = (A^T)^T \cdot A^T$$

$$= A \cdot A^T \quad \left[\because (A^T)^T = A\right]$$

$\therefore A \cdot A^T$ —— symmetric.

$A^T A$ —— symmetric (prove it!)

prob: Show that the matrix $B^T A B$ is symmetric or skew symmetric according as A is symmetric or skew-symmetric.

Sol^n.

$$(B^T A B)^T = B^T A^T \cdot (B^T)^T$$
$$= B^T A^T B$$
$$= B^T A B \text{ if } A^T = A$$

i.e. A is symm.

$\therefore$ $B^T A B$ is symmetric if A is symm.

$$(B^T A B)^T = B^T \cdot A^T \cdot (B^T)^T$$
$$= B^T \cdot A^T \cdot B$$
$$= -B^T A B \text{ if } A^T = -A$$

$\therefore$ $B^T A B$ is skew-symmetric if A is skew-symmetric.

where

$P = \frac{1}{2}(A + A^T)$

$Q = \frac{1}{2}(A - A^T)$

$P^T = \frac{1}{2}(A + A^T)^T$

$= \frac{1}{2}(A^T + (A^T)^T)$

$= \frac{1}{2}(A^T + A)$

$= \frac{1}{2}(A + A^T)$

$= P$

$\therefore P$ — symmetric

Prob: Show that every square matrix is uniquely expressible as the sum of symmetric and skew symmetric matrices.

Sol^n: Let A — be a square matrix.

$$A = \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T$$

$$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= \underset{(\text{say})}{P + Q} \quad \text{—— (1)}$$

Unique:

Let $A = R + S$ — (ii)

where R is symmetric and S is skew-symm.

(ii) $\Rightarrow$

$A^T = (R+S)^T$

$\Rightarrow A^T = R^T + S^T$

$A^T = R - S$ — (iii)

(ii) + (iii)

$A + A^T = 2R \Rightarrow R = \frac{1}{2}(A + A^T)$
$= P$

$Q^T = \frac{1}{2}(A - A^T)^T$

$= \frac{1}{2}(A^T - (A^T)^T)$

$= \frac{1}{2}(A^T - A)$

$= -\underbrace{\frac{1}{2}(A - A^T)}$

$= -Q$

$\therefore Q$ is skew-symmetric.

II − III

$$A - A^T = \cancel{R} + S - (\cancel{R} - S)$$

$$A - A^T = S + S = 2S$$

$$\Rightarrow S = \frac{1}{2}(A - A^T)$$

$$= \varphi.$$

Prob: Show that all the integral powers of a symmetric matrix is symmetric.

→ Let A is a symmetric matrix.

$\therefore A^T = A$ —— (1)

$$(A^n)^T = (A \cdot A \cdot A \cdots n \text{ times})^T$$

$$= A^T \cdot A^T \cdot A^T \cdot A^T \cdots n \text{ times}$$

$$= A \cdot A \cdot A \cdots n \text{ times}$$

$$= \underline{A^n}.$$

Prob: Show that the odd integral powers of a skew symmetric matrix is a skew-symmetric matrix while the even integral powers are symmetric matrices.

Sol^n: Let A be a skew-symmetric matrix.

$$\therefore A^T = -A \quad \text{— (1)}$$

$$(A^m)^T = (A \cdot A \cdot \ldots m \text{ times})^T$$

$$= A^T \cdot A^T \ldots m \text{ times}$$

$$= (-A)(-A) \ldots m \text{ times}$$

$$= (-1)^m \cdot A^m$$

(i) m is odd

$$(A^m)^T = -A^m$$

$\therefore A^m$ is skew-symmetric.

(ii) m is even

$$(A^m)^T = A^m$$

$\therefore A^m$ is symmetric.

Prob: If U and V are two symmetric matrices, then show that UVU is also symmetric. Is UV symmetric always? Explain and illustrate by an example.

$\because$ U and V are symmetric.

$\therefore U^T = U \;\&\; V^T = V \longrightarrow (1)$

$$(UVU)^T = U^T \cdot V^T \cdot U^T$$
$$= UVU$$

$\therefore$ UVU is symmetric.

NO.

$$(UV)^T = V^T \cdot U^T$$
$$= V \cdot U$$

$$= UV \text{ if } VU = UV.$$

UV is symmetric if $UV = VU$.

$$(UV)^T = V^T U^T$$
$$= VU$$
$$\neq UV \quad [\text{By ex.}]$$

$\therefore$ UV isn't symmetric.

Ex:

(Symm.) $U = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $\quad V = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ Symm.

$$UV = \begin{bmatrix} 8 & 11 \\ 13 & 18 \end{bmatrix}$$

$$VU = \begin{bmatrix} 8 & 13 \\ 11 & 18 \end{bmatrix}$$

$UV \neq VU$.