KGS IAS

Maths Optional

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sol^n:

$$Z \cdot x = \int x \cdot \left(\frac{-1}{x}\right) dx + \text{const.}$$

$$\Rightarrow Z x = -x + c_1$$

$$\Rightarrow x Z + x = c_1$$

$$\Rightarrow x \frac{dy}{dx} + x = c_1$$

$$\Rightarrow \frac{dy}{dx} + 1 = \frac{c_1}{x}$$

$$\Rightarrow dy = \left(\frac{c_1}{x} - 1\right) dx$$

HW prob: $x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 1 = 0$

Let $Z = \dfrac{dy}{dx}$

$$x \frac{dZ}{dx} + Z + 1 = 0$$

$$\Rightarrow \frac{dZ}{dx} + \frac{1}{x} dZ = -\frac{1}{x}$$

$$IF = e^{\int \frac{1}{x} dx} = e^{\ln x}$$

$$= x$$

Equations reducible to Linear form:

(1) $f'(y)\frac{dy}{dx} + P \cdot f(y) = Q$ — (1)

P, Q are fn. of x.

put $t = f(y)$

$$\frac{dt}{dx} = f'(y)\frac{dy}{dx}$$

(1) $\Rightarrow$

$$\frac{dt}{dx} + Pt = Q$$

Integrate,

$$y = c_1 \log x - x + c_2$$

$$\Rightarrow \boxed{y + x = c_1 \log x + c_2}$$

(iii) Bernoulli's eqn.

$$\frac{dy}{dx} + Py = Q y^n \quad \text{--- (iii)}$$

P & Q are fn. of 'x'.

$n \neq 0, 1$

(iii) $\Rightarrow$ $y^{-n}\frac{dy}{dx} + P\cdot y^{-n+1} = Q$ — (iv)

Put $t = y^{-n+1}$

(ii) $f'(x)\frac{dx}{dy} + P\cdot f(x) = Q$. — (ii)

P & Q are fn. of 'y'.

Put $t = f(x)$

$$\frac{dt}{dy} = f'(x)\cdot\frac{dx}{dy}$$

(ii) $\Rightarrow$

$$\frac{dt}{dy} + Pt = Q.$$

Note: $\frac{dx}{dy} + p\,x = \phi\cdot x^n$

p, ϕ are fn. of y.

$x^{-n}\frac{dx}{dy} + p\,x^{-n+1} = \phi$

Put $t = x^{-n+1}$

$\frac{dt}{dy} = (1-n)x^{-n}\cdot\frac{dx}{dy}$

$x^{-n}\frac{dx}{dy} = \frac{1}{1-n}\frac{dt}{dy}$

$\frac{dt}{dx} = (1-n)\cdot y^{-n}\cdot\frac{dy}{dx}$

$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\cdot\frac{dt}{dx}$

(IV) $\Longrightarrow$

$\frac{1}{1-n}\frac{dt}{dx} + p\cdot t = \phi$

$\frac{dt}{dx} + (1-n)p\,t = (1-n)\phi$

Compare:

$$\boxed{\frac{dt}{dx} + Pt = Q}$$

$P = 2x, \quad Q = x^3.$

$I.f = e^{\int 2x\,dx} = e^{x^2}$

Soln. is

$$t \cdot e^{x^2} = \int e^{x^2} \cdot x^3\,dx + \text{const} \quad (1)$$

$$\int e^{x^2} \cdot x^3\,dx = \frac{1}{2}\int z e^z\,dz$$

Put $z = x^2$

Prob: $\sec^2 y \cdot \frac{dy}{dx} + 2x \tan y = x^3$ — (1)

Put $t = \tan y$

$$\frac{dt}{dx} = \sec^2 y \frac{dy}{dx}$$

(1) $\Rightarrow$ $\frac{dt}{dx} + 2xt = x^3$

$$t\cdot e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + \text{const}$$

$$t = \frac{1}{2}(x^2-1) + c\cdot e^{-x^2}$$

$$\tan y = \frac{1}{2}(x^2-1) + c\cdot e^{-x^2}$$

$$= \frac{1}{2}\left[\int z e^{z}\, dz\right]$$

$$= \frac{1}{2}\left[z\int e^{z} - \int e^{z}\, dz\right]$$

$$= \frac{1}{2}\left[z e^{z} - e^{z}\right]$$

$$= \frac{1}{2} e^{z}(z-1)$$

$$= \frac{1}{2} e^{x^2}(x^2-1)$$

(1) $\Rightarrow$

$$-\frac{dt}{dx}+\frac{1}{x}\cdot t=\frac{1}{x^2}$$

$$\Rightarrow \frac{dt}{dx}-\frac{1}{x}t=-\frac{1}{x^2}$$

$$IF=e^{-\int\frac{1}{x}dx}=e^{-\log x}$$
$$=e^{\log\frac{1}{x}}=\frac{1}{x}.$$

Sol^n.

$$t\cdot\frac{1}{x}=\int\frac{1}{x}\left(-\frac{1}{x^2}\right)dx+\text{const}$$

$$\frac{t}{x}=+\frac{x^{-2}}{+2}+\text{const}$$

Prob:

$$\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$$

$$\Rightarrow e^{-y}\frac{dy}{dx}+e^{-y}\cdot\frac{1}{x}=\frac{1}{x^2} \quad \text{--- (1)}$$

Put $t=e^{-y}$

$$\frac{dt}{dx}=-e^{-y}\frac{dy}{dx}$$

$$e^{-y}\frac{dy}{dx}=-\frac{dt}{dx}$$

$$\Rightarrow x^{-2}\frac{dx}{dy} - x^{-1}y = y^3 \quad \text{——} (1)$$

$$\text{put } t = -\frac{1}{x}$$

$$\frac{dt}{dy} = \frac{1}{x^2}\cdot\frac{dx}{dy}$$

$$\therefore (1) \Rightarrow$$

$$\frac{dt}{dy} + ty = y^3$$

$$IF = e^{\int y\,dy} = e^{\frac{y^2}{2}}$$

$$\frac{t}{x} = \frac{1}{2x^2} + C$$

$$e^{-y} = \frac{1}{2x} + Cx$$

Prob: $\frac{dy}{dx}(x^2y^3 + xy) = 1$

$$\Rightarrow x^2y^3 + xy = \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3$$

$$2\int e^{z} z \, dz$$

$$= 2\left[z e^{z} - \int e^{z} dz\right]$$

$$= 2\left[z e^{z} - e^{z}\right]$$

$$= 2 e^{z}\left[z - 1\right]$$

$$\therefore (11) \Rightarrow$$

$$t \cdot e^{\frac{y^2}{2}} = 2 e^{\frac{y^2}{2}}\left[\frac{y^2}{2} - 1\right] + \text{Const}$$

$$\Rightarrow t = 2\left[\frac{y^2}{2} - 1\right] + c \cdot e^{-\frac{y^2}{2}}$$

$$\Rightarrow -\frac{1}{x} = \left[y^2 - 2\right] + c \cdot e^{-\frac{y^2}{2}}.$$

$$t \cdot e^{\frac{y^2}{2}} = \int \cdot e^{\frac{y^2}{2}} \cdot y^3 dy + \text{Const.} \quad (11)$$

$$\int e^{\frac{y^2}{2}} y^3 dy$$

$$z = \frac{y^2}{2}$$

$$\frac{dz}{dy} = y$$

$$\Rightarrow dz = y\,dy$$

$$IF = e^{-2\int \frac{1}{x+1}dx}$$

$$= e^{-2\log(x+1)}$$

$$= e^{\log\frac{1}{(x+1)^2}}$$

$$= \frac{1}{(x+1)^2}$$

$\therefore$ soln is

$$t \cdot \frac{1}{(x+1)^2} = \int \frac{1}{(x+1)^2} \times (x+1)^3 dx + \text{const}$$

$$\Rightarrow \frac{t}{(x+1)^2} = \left(\frac{x^2}{2} + x\right) + c$$

$$\Rightarrow \frac{\sin(y^2)}{(x+1)^2} = \left(\frac{x^2}{2} + x\right) + c.$$

prob:

$$2y\cos(y^2)\frac{dy}{dx} - \frac{2\sin(y^2)}{x+1} = (x+1)^3 \quad \text{--- (1)}$$

put $t = \sin(y^2)$

$$\frac{dt}{dx} = 2\cos(y^2)\cdot y\frac{dy}{dx}$$

$$(1) \Rightarrow \frac{dt}{dx} - \frac{2}{x+1}\cdot t = (x+1)^3$$

$$\Rightarrow y^{-2}\frac{dy}{dx}+\frac{1}{x}\cdot y^{-1}=x \quad -(1)$$

put $t=y^{-1}$

$$\frac{dt}{dx}=-\frac{1}{y^2}\frac{dy}{dx}$$

$$-\frac{dt}{dx}=y^{-2}\frac{dy}{dx}$$

$$(1)\Rightarrow -\frac{dt}{dx}+\frac{t}{x}=x$$

$$\Rightarrow \frac{dt}{dx}-\frac{t}{x}=-x$$

Prob: Find the eqn. of the curve which passes through the point $(1,-1)$ and satisfies the diff. eqn.

$$x\frac{dy}{dx}+y=x^2y^2$$

Sol^n

$$\frac{dy}{dx}+\frac{1}{x}\cdot y=xy^2$$

At $x=1,\ y=-1$

$$\frac{1}{1\times(-1)} = -1+c$$

$$\Rightarrow -1 = -1+c$$

$$\Rightarrow \underline{c=0}$$

$$\frac{1}{xy} = -x$$

$$\Rightarrow 1 = -x^2 y$$

$$\Rightarrow \boxed{x^2 y = -1}$$

$$\text{If} = e^{-\int \frac{1}{x} dx}$$

$$= e^{-\log x}$$

$$= e^{\log \frac{1}{x}} = \frac{1}{x}$$

Solⁿ. is

$$t \cdot \frac{1}{x} = \int \frac{1}{x}(-x)\,dx + \text{const}$$

$$t \cdot \frac{1}{x} = -x + \text{const}$$

$$\Rightarrow \frac{1}{xy} = -x + c$$

(1) $\Rightarrow$

$$-\frac{dt}{dx} + t \cdot \frac{1}{x} = \frac{1}{x^2}$$

$$\Rightarrow \frac{dt}{dx} - \frac{1}{x} \times t = -\frac{1}{x^2}$$

$$IF = e^{-\int \frac{1}{x} dx}$$

$$= e^{-\log x} = e^{\log \frac{1}{x}}$$

$$= \frac{1}{x}.$$

Sol^n

$$t \cdot \frac{1}{x} = \int \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) dx + \text{Const}$$

Prob: $$\frac{dz}{dx} + \frac{z}{x} \log z = \frac{z}{x^2} (\log z)^2$$

$$\Rightarrow \frac{1}{z \cdot (\log z)^2} \cdot \frac{dz}{dx} + \frac{1}{\log z} \cdot \frac{1}{x} = \frac{1}{x^2} \quad \text{— (1)}$$

Put $t = \frac{1}{\log z}$

$$\frac{dt}{dx} = -(\log)^{-2} \cdot \frac{1}{z} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{z (\log z)^2}$$

$$\frac{t}{x} = -\int x^{-3}\,dx + \text{const.}$$

$$\Rightarrow \frac{t}{x} = + \frac{x^{-2}}{+2} + \text{const}$$

$$\Rightarrow \frac{t}{x} = \frac{1}{2x^2} + \text{const.}$$

$$\Rightarrow \frac{1}{x \log z} = \frac{1}{2x^2} + C$$