$\rightarrow$

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$A \cdot A^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^{\dagger} \cdot A = I$$

$$i^2 = -1$$

$$\longrightarrow \quad A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$A^T = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$A^{\dagger} = \overline{(A^T)} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

Prob: If A and B are square matrices and are also orthogonal then AB & BA are orthogonal.

Sol^n: $\because$ A and B are orthogonal

$$\left.\begin{array}{l} \therefore AA^T = A^T A = I \\ BB^T = B^T B = I \end{array}\right\} \text{———} (1)$$

$$(AB)^T (AB) = (B^T A^T)(AB)$$

$$= B^T [A^T (AB)]$$

$$= B^T [(A^T A) B]$$

$$= B^T [I \cdot B] \quad \text{using } (1)$$

$$= B^T \cdot B$$

$$= I \quad (\text{using } (1))$$

$\therefore$ AB is orthogonal.

(BA) —— " —— HW

$$= B^{\theta}\left(A^{\theta}(AB)\right)$$

$$= B^{\theta}\left((A^{\theta}\cdot A)B\right)$$

$$= B^{\theta}(I\cdot B) \quad \text{using (1)}$$

$$= B^{\theta}\cdot B$$

$$= I \quad (\text{using (1)}$$

$\therefore$ AB is unitary.

BA ———— $\boxed{\text{HW}}$

prob: If A and B are n-rowed unitary matrices then AB & BA are also unitary matrices.

Sol^n: $\because$ A and B are unitary matrices.

$$\left.\begin{aligned} \therefore\ A^{\theta}A &= A\cdot A^{\theta} = I \\ B^{\theta}\cdot B &= B\cdot B^{\theta} = I \end{aligned}\right\} \text{— (1)}$$

$$(AB)^{\theta}\cdot(AB) = (B^{\theta}\cdot A^{\theta})(AB)$$

$\because A$ is real $\therefore \bar{A} = A$

$A^{\theta} = (\bar{A})^T$

$= (A)^T$

Prob: A real matrix is unitary $\iff$ it is orthogonal.

Solⁿ: Let A be a real square matrix and it is unitary.

A is unitary $\iff A^{\theta} \cdot A = I$

$\iff A^T \cdot A = I$

$\iff A$ is orthogonal.

$$(P^{-1})^T \cdot P^{-1} = (P^T)^{-1} \cdot P^{-1}$$

$$= [P \cdot P^T]^{-1}$$

$$= [I]^{-1}$$

using (1)

$$= I$$

$\therefore$ P^{-1} is orthogonal.

Prob: If P is orthogonal then P^T and P^{-1} are also orthogonal.

Solⁿ: $\because$ P is orthogonal

$$\therefore \quad P^T P = P \cdot P^T = I \longrightarrow (1)$$

$$(P^T)^T \cdot P^T = P \cdot P^T = I \quad (\text{using } (1))$$

$\therefore$ P^T is orthogonal.

Prob: If P is unitary then $\overline{P}, P^T, P^\theta$ and P^{-1} are also unitary.

Solⁿ: $\because$ P is unitary

$\therefore P^\theta \cdot P = P \cdot P^\theta = I \quad \text{—— (1)}$

$$(\overline{P})^\theta \cdot \overline{P} = \left(\overline{\overline{P}}\right)^T \cdot \overline{P}$$

$$= P^T \cdot \overline{P}$$

$$= \overline{\overline{(P^T)}} \cdot \overline{P}$$

$$= \overline{\left(\overline{P^T}\right)} \cdot \overline{P}$$

$$= \overline{\left((\overline{P})^T\right)} \cdot \overline{P}$$

$$= \overline{P^\theta} \cdot \overline{P}$$

$$= \overline{P^\theta \cdot P} \quad \left[\because \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}\right]$$

$$= \overline{I} \quad [\text{using (1)}]$$

$$= I$$

$\therefore \overline{P}$ is unitary.

P^θ, P^{-1} — unitary

HW

$(P^T)^\theta \cdot P^T$

$= \left(\overline{P^T}\right)^T \cdot P^T$

$= \left(\left(\overline{P}\right)^T\right)^T \cdot P^T$

$= (P^\theta)^T \cdot P^T$

$= (P \cdot P^\theta)^T$ [using (1)]

$= (I)^T = I$

$\therefore P^T$ is unitary.

Prob: A real skew-symmetric matrix A satisfies the relation $A^2 + I = 0$, where I is the identity matrix, show that A is orthogonal.

Sol^n: $\because$ A is skew-symmetric Matrix.

$\therefore A^T = -A$ —— (i)

Also $A^2 + I = 0$

$\Rightarrow A^2 = -I \Rightarrow I = -A^2$ —— (ii)

$A^T \cdot A = (-A) \cdot A$ using (i)

$= -A^2$

$= I$ (using (ii))

$\therefore$ A is orthogonal.

Prob: Show that if A is Hermitian and P is unitary then $P^{-1}AP$ is Hermitian.

Sol^n: $\because$ A is Hermitian.

$$\therefore\ A^{\theta} = A \quad \text{——— (i)}$$

$\because$ P is unitary.

$$\therefore\ P^{\theta} \cdot P = P \cdot P^{\theta} = I \quad \text{——— (ii)}$$

$$\Rightarrow\ P^{-1} = P^{\theta} \quad \text{——— (iii)}$$

$$(P^{-1}AP)^{\theta} = P^{\theta} \cdot A^{\theta} \cdot (P^{-1})^{\theta}$$

$$= P^{\theta} \cdot A^{\theta} \cdot (P^{\theta})^{\theta} \quad \text{using (iii)}$$

$$= P^{\theta} \cdot A \cdot P \quad \text{(using (i))}$$

$$= P^{-1} \cdot A \cdot P \quad \text{(using (iii))}$$

$\therefore$ $P^{-1}AP$ is Hermitian.

Rank of a Matrix

Submatrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 2 & 5 \\ 2 & 2 & 3 & 0 & 1 \\ 3 & 3 & 4 & -2 & -1 \\ -4 & -1 & 1 & -3 & 4 \end{bmatrix}_{4\times 5}$$

B is called a submatrix of A.

Ex: Delete R_1, C_1, C_2

$$C = \begin{bmatrix} 3 & 0 & 1 \\ 4 & -2 & -1 \\ 1 & -3 & 4 \end{bmatrix}$$

C is a submatrix of A.

Delete R_1, R_2 & C_3

$$B = \begin{bmatrix} 3 & 3 & -2 & -1 \\ -4 & -1 & -3 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 2 & 3 \\ 3 & -1 & 4 \end{vmatrix} \text{ — Minor}$$

Note: A matrix is a sub-matrix of itself.

Minors: Determinant of a square submatrix is called a minor.

Ex: $A = \begin{bmatrix} 1 & 5 & 0 & -1 & 1 \\ 2 & 2 & 3 & 1 & 0 \\ 3 & -1 & 4 & 2 & 2 \\ 4 & 2 & -1 & -2 & -1 \end{bmatrix}_{4\times 5}$

$$\begin{vmatrix} 1 & 5 \\ 2 & 2 \end{vmatrix} = 2 - 10 = \underline{-8} \text{ — Minor}$$

Rank of a matrix: A number r is said to be the rank of a matrix A iff

(i) $\exists$ at least one minor of order r which is non-zero.

(ii) All $(r+1)$-rowed minors should be zero.

We denote rank of a matrix A by $r(A)$ or $\rho(A)$

Ex:

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 2 \\ 3 & 1 \end{bmatrix}_{3\times 2}$$

$$\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} = 2+8$$

$$= 10 \neq 0$$

Rank of $A = 2$.

Expanding through C_1

$$|A| = 1 \cdot \begin{vmatrix} 8 & 6 \\ -5 & 3 \end{vmatrix}$$

$$= 24 + 30 = 54 \neq 0$$

$\therefore \rho(A) = 3$

Ex:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix}_{3\times 3}$$

$$|A| = \begin{vmatrix} 1 & 3 & 1 \\ -2 & 2 & 4 \\ 2 & 1 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 1 \\ 0 & 8 & 6 \\ 0 & -5 & 3 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

(IV) $r(I_n) = n$

(V) $A_{m\times n}$ — If A has a non-zero minor of order r.

$r(A) \geq r$.

(VI) $r(A) = r(A^T)$

(VII) $r(O) = 0$ — No.

matrix

Note (I) Rank of a matrix is unique.

(II) $A_{m\times n}$ — matrix.

$r(A) \leq \min\{m, n\}$.

(III) A is a non-singular matrix of order n.

$|A| \neq 0$

$\therefore r(A) = n$.

Ex:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{4\times 3}$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$r(A) = 1$.

VIII Rank of a non-zero matrix is ≥ 1.

IX Rank of a matrix, whose elements are '1' only, is 1.

X $A_{m\times n}$

Every r-rowed minor is zero.

$r(A) < r$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ -3 & 12 & 15 \\ -1 & 2 & 0 \end{bmatrix}$$

$R_2(3)$
$(R_2 \to 3R_2)$

→ Elementary operation (trans-formation) of a matrix

		Notation
(I)	$R_i \longleftrightarrow R_j$	R_{ij}
(II)	$R_i \to kR_i$	$R_i(k)$
(III)	$R_i \to R_i + kR_j$	$R_{ij}(k)$

$$A = \begin{bmatrix} 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$$

Echelon form

$$B = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Echelon form

Echelon matrix: A matrix A is said to be in echelon form if

(I) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

(II) The elements of the last row(s) may be all zeros.

$$C = \begin{bmatrix} 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon form.

$R(C) = 2 =$ No. of non-zero rows.

<u>Note</u>: The rank of a matrix in echelon form = No. of non-zero rows.