Maths Optional

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Sol^n: Let $A = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $B = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\rightarrow A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

Echelon forms of A & B have same non-zero rows.

Prob: Let $U = \text{Span}(u_1, u_2, u_3)$ and $W = \text{Span}(v_1, v_2)$ be two subspaces of $\mathbb{R}^4$ where $u_1 = (1, 2, -1, 3)$, $u_2 = (2, 4, 1, -2)$, $u_3 = (3, 6, 3, -7)$, $v_1 = (1, 2, -4, 11)$, $v_2 = (2, 4, -5, 14)$; Show that $U = W$.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \end{bmatrix} \; R_1 \to R_1 + R_2$$

$\therefore$ Row space of A = Row space of B

i.e. $\text{Span}\{u_1, u_2, u_3\} = \text{Span}\{v_1, v_2\}$

i.e. $U = W$

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix} \; R_2 \to R_2 - 2R_1$$

Consider $A = \begin{bmatrix} 1 & -4 & -2 & 1 \\ 1 & -3 & -1 & 2 \\ 3 & -8 & -2 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis of $W = \{(1,-4,-2,1), (0,1,1,1)\}$

$\operatorname{Dim} W = 2$

prob: Find a basis and dimension of the subspace W of $\mathbb{R}^4$ generated by $(1,-4,-2,1), (1,-3,-1,2), (3,-8,-2,7)$. Also extend the basis of W to a basis of $\mathbb{R}^4$.

$(0,0,1,0) \notin L(S)$

let $S_1 = \{(1,-4,-2,1), (0,1,1,1), (0,0,1,0)\}$ is L.I

$L(S_1) = \{a(1,-4,-2,1) + b(0,1,1,1) + c(0,0,1,0) : a,b,c \in \mathbb{R}\}$

$= \{(a, -4a+b, -2a+b+c, a+b) : a,b,c \in \mathbb{R}\}$

let $S = \{(1,-4,-2,1), (0,1,1,1)\}$

$L(S) = \{a(1,-4,-2,1) + b(0,1,1,1) : a,b \in \mathbb{R}\}$

$= \{(a, -4a+b, -2a+b, a+b) : a,b \in \mathbb{R}\}$

$$\begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow$ Echelon form

$\therefore$ Row vectors are L.I.

$(0, 0, 0, 1) \notin L(S_1)$

Let $S_2 = \{(1, -4, -2, 1), (0, 1, 1, 1), (0, 0, 1, 0), (0, 0, 0, 1)\}$

S_2 is L.I

S_2 has 4 elements and $\operatorname{Dim} \mathbb{R}^4 = 4$

$\therefore$ S_2 is a basis of $\mathbb{R}^4$.

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 4 & -2 & -2 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 0 & -\frac{10}{3} & -10 \end{bmatrix} \quad R_3 \rightarrow R_3 - \frac{4}{3}R_1$$

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 1 & 3 \\ 0 & -10 & -30 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow \frac{1}{2}R_2 \\ R_3 \rightarrow 3R_3 \end{matrix}$$

Prob: Let S be the space generated by the vectors $\{(0,2,6), (3,1,6), (4,-2,-2)\}$. What is dimension of the space S? Find basis for S.

Sol^n: Let $A = \begin{bmatrix} 0 & 2 & 6 \\ 3 & 1 & 6 \\ 4 & -2 & -2 \end{bmatrix}$

Coset: Let W be any subspace of a vector space V(F).

$$W+\alpha = \{w+\alpha : w \in W\}$$

$\alpha \in V$

→ Right coset of W in V generated by α.

$$\alpha + W = \{\alpha + w : w \in W\}$$

→ left coset of W in V.

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 10R_2$$

Basis of $S = \{(3,1,6),(0,1,3)\}$

Dim $S = 2$

(iii) $\alpha = 0 \in V$

$W + \alpha = W + 0$

$= W$

$\downarrow$

coset in V

(iv) $\underline{\alpha \in W} \Rightarrow \underline{W + \alpha = W}$

Note (i) $W+\alpha$, $\alpha + W$

are subsets of V.

(ii) '+' operation — commutative in V.

$\therefore W + \alpha = \alpha + W$

Right coset and Left coset same.

$\boxed{W+\alpha}$ — a coset of W in V generated by α.

vector space over F.

Internal composition and external composition are defined as:

(i) $(W+\alpha)+(W+\beta) = W+(\alpha+\beta)$

(ii) $a(W+\alpha) = W+a\alpha$

$\forall \alpha, \beta \in V, \forall a \in F$.

(5) $W+\alpha = W+\beta$

$\iff \alpha-\beta \in W$

Th: If W is any subspace of a vector space $V(F)$, then the set $\frac{V}{W}$ of all cosets $W+\alpha$, where $\alpha \in V$ i.e.,

$\frac{V}{W} = \{W+\alpha : \alpha \in V\}$ is a

$$= (W+\alpha) + [W + (\beta+\gamma)]$$

$$= W + [\alpha + (\beta+\gamma)]$$

$$= W + [(\alpha+\beta)+\gamma]$$

$\because \alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$ in V

$$= [W+(\alpha+\beta)] + (W+\gamma)$$

$$= [(W+\alpha)+(W+\beta)] + (W+\gamma)$$

II $\left(\frac{V}{W}, +\right)$ is an abelian group.

1. $(W+\alpha)+(W+\beta)$

$$= W+(\alpha+\beta) \in V/W$$

$\because \alpha, \beta \in V$

$\therefore \alpha+\beta \in V$

2. $W+\alpha,\ W+\beta,\ W+\gamma \in \frac{V}{W}$

$$(W+\alpha)+[(W+\beta)+(W+\gamma)]$$

$\therefore W+0=W$ is the identity element of $\frac{V}{W}$.

(IV) Let $W+\alpha \in V/W$, $\alpha \in V$

$\because \alpha \in V \Rightarrow -\alpha \in W$

$\therefore W+(-\alpha) \in \frac{V}{W}$

Now $(W+\alpha)+(W+(-\alpha))$

$= W+(\alpha+(-\alpha))$

$= W+0 = W$

(III) Let $W+\alpha \in \frac{V}{W}$.

Also, $W+0=W \in \frac{V}{W}$

Now $(W+0)+(W+\alpha)$

$= W+(0+\alpha)$

$= W+\alpha$ $\quad \because 0+\alpha=\alpha$ in V

Similarly, we can prove

$(W+\alpha)+(W+0) = W+\alpha$

(v) $W+\alpha,\ W+\beta \in V/W$

$$(W+\alpha)+(W+\beta) = W+(\alpha+\beta)$$
$$= W+(\beta+\alpha)$$

As $\alpha+\beta=\beta+\alpha$ in V

$$= (W+\beta)+(W+\alpha)$$

As $\alpha+(-\alpha)=0 \in V$

Similarly, we can prove

$$[W+(-\alpha)]+(W+\alpha) = W$$

$\therefore$ $W+(-\alpha)=W-\alpha$ is the Inverse of $W+\alpha$ in V/W.

II Let $a,b \in F$, $W+\alpha, W+\beta \in V/W$

(i) $(a+b)(W+\alpha)$
$$= W+(a+b)\alpha$$

Thus $\frac{V}{W}$ is a vect-
or over the field F.

$\frac{V}{W}(F)$ — quotient space of V relative to W.

$= W + (a\alpha + b\alpha)$

As $(a+b)\cdot\alpha = a\alpha + b\alpha$ in V

$= (W + a\alpha) + (W + b\alpha)$

$= a(W+\alpha) + b(W+\alpha)$

(Prove it!)

(II) $a\cdot[(W+\alpha) + (W+\beta)]$

$=$

(III)

(IV)

Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ be a basis of W.

$\therefore \operatorname{Dim} W = m$

$\therefore S$ is a L.I. subset of W.

Also $S \subseteq W \subseteq V$

$\therefore S \subseteq V$

$\therefore S$ is a L.I. subset of V.

$\therefore$ It can be extended to form the basis of V.

Th: If W be a subspace of a FDVS $V(F)$ then

$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W.$$

Proof: W is a subspace of a FDVS $V(F)$.

$\therefore$ W is also finite dimensional.

For this we will show that

$$S_2 = \{ w+\beta_1, w+\beta_2, \dots, w+\beta_l \}$$

is a basis of V/W.

<u>S_2 is L.I.</u>

Let

$$b_1(w+\beta_1) + b_2(w+\beta_2) + \dots + b_l(w+\beta_l) = w = w+0 \quad (1)$$

$b_i \in F$.

Let $S_1 = \{ \alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_l \}$ be a basis of V.

$\therefore \operatorname{Dim} V = m+l$.

Now $\operatorname{Dim} V - \operatorname{Dim} W$

$= m+l-m$

$= l$

<u>We claim that</u>: $\dim\left(\frac{V}{W}\right) = l$.

S is a basis of W

$\therefore b_1\beta_1 + b_2\beta_2 + \cdots + b_l\beta_l$

$= \underbrace{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m}_{\text{L.C. of elements of } S}$

$a_i \in F$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_l\beta_l - a_1\alpha_1 - a_2\alpha_2 - \cdots - a_m\alpha_m = 0$

(1)

$(W + b_1\beta_1) + (W + b_2\beta_2) + \cdots + (W + b_l\beta_l) = W$

$\Rightarrow W + (b_1\beta_1 + b_2\beta_2 + \cdots + b_l\beta_l) = W + 0$

$\Rightarrow (b_1\beta_1 + b_2\beta_2 + \cdots + b_l\beta_l) - 0 \in W$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_l\beta_l \in W$

W.K.T. $L(S_2) \subseteq V/W$

let $W+\alpha \in V/W$,

$\alpha \in V(F)$

$\therefore S_1$ is a basis

of V.

$\therefore \exists\, c_1, c_2, \dots c_m,$

$d_1, d_2 \dots d_l \in F$

s.t

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_m\alpha_m +$

$d_1\beta_1 + d_2\beta_2 + \dots + d_l\beta_l$.

$\because S_1 = \{\alpha_1, \alpha_2, \dots \alpha_m, \beta_1, \beta_2, \dots \beta_l\}$ is a basis of V.

$\therefore S_1$ is L.I.

(II) $\Rightarrow$

$a_1 = a_2 = \dots = b_1 = b_2 \dots = b_l = 0$

i.e. (I) $\Rightarrow b_1 = b_2 = \dots = b_l = 0$

$\therefore S_2$ is L.I

(iii) $\Rightarrow$

$$W+\alpha = W + (\gamma + d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l)$$

$$= (W+\gamma) + [W + (d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l)]$$

$$= W + [W + (d_1\beta_1 + \cdots + d_l\beta_l)]$$

$$= W + (d_1\beta_1 + \cdots + d_l\beta_l) \quad \because \gamma \in W \Rightarrow W+\gamma = W$$

$$\Rightarrow W+\alpha = W + (c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m + d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l) \text{ — (iii)}$$

$\because S = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a basis of W.

$\therefore c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m \in W$

def $\gamma = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m$

$\therefore \gamma \in W$

And so

$$\frac{V}{W} = L(S_2)$$

Thus, S_2 is a basis of

$$\frac{V}{W}$$

$$\therefore \operatorname{Dim} \frac{V}{W} = l$$
$$= \dim V - \dim W$$

$$\therefore w+\alpha = (w + d_1\beta_1) + (w + d_2\beta_2) + \cdots + (w + d_l\beta_l)$$

$$= \underbrace{d_1(w+\beta_1) + d_2(w+\beta_2) + \cdots + d_l(w+\beta_l)}_{\text{L.C. of elements of } S_2}$$

$$\therefore w+\alpha \in L(S_2)$$

$$\therefore \frac{V}{W} \subseteq L(S_2)$$

Prob: Determine $\dim\left(\frac{V}{W}\right)$,

where $V = \mathbb{C}(\mathbb{R})$,

$W = \mathbb{R}(\mathbb{R})$.

Sol^n. Basis of $V = \{1, i\}$.

Basis of $W = \{1\}$.

$\therefore$ $\text{Dim } V = 2$

$\text{Dim } W = 1$

$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W$$
$$= 2 - 1$$
$$= 1.$$

$P_4 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 : a_i \in \mathbb{R}\}$

Basis of $P_4 = \{1, x, x^2, x^3, x^4\}$

$\text{Dim } P_4 = 5$

$P_2 = \{a_0 + a_1 x + a_2 x^2 : a_i \in \mathbb{R}\}$

Basis of $P_2 = \{1, x, x^2\}$.

$\therefore \text{Dim of } P_2 = 3$

<u>Prob</u>: Let P_n denotes the vector space of all polynomials of degree $\leq n$. Exhibit a basis for $\frac{P_4}{P_2}$, hence verify that

$$\dim\left(\frac{P_4}{P_2}\right) = \dim P_4 - \dim P_2.$$

$$\boxed{\begin{aligned} &a_3x^3 + a_4x^4 + P_2 \\ &= (a_3x^3 + P_2) + (a_4x^4 + P_2) \\ &= a_3(x^3 + P_2) + a_4(x^4 + P_2) \end{aligned}}$$

Basis of $\frac{P_4}{P_2} = \{P_2 + x^3, P_2 + x^4\}$

$\text{Dim } \frac{P_4}{P_2} = 2$

$\text{Dim } P_4 - \text{Dim } P_2 = 5 - 3 = 2$

$$\frac{P_4}{P_2} = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + P_2 : a_i \in \mathbb{R}\}$$

$$= \{a_3x^3 + a_4x^4 + P_2 : a_3, a_4 \in \mathbb{R}\}$$

$a_0 + a_1x + a_2x^2 \in P_2$

$\therefore (a_0 + a_1x + a_2x^2) + P_2$

$= P_2$

Sol^n: $\text{Dim } A = 2$,

$\{(1,0,0),(0,1,0)\}$ is a basis of A.

$\text{Dim } B = 2$,

$\{(0,1,0),(0,0,1)\}$ is a basis of B.

$\text{Dim}(A \cap B) = 1$,

$\{(0,1,0)\}$ is a basis of $A \cap B$

Prob: Let $A = \{(x,y,0) : x, y \in \mathbb{R}\}$

$B = \{(0,y,z) : y, z \in \mathbb{R}\}$

be two subspace of $\mathbb{R}^3$.

Verify that

$$\dim\left(\frac{A+B}{A}\right) = \dim\left(\frac{B}{A \cap B}\right)$$

$$\operatorname{Dim}\left(\frac{B}{A\cap B}\right) = \operatorname{Dim} B - \operatorname{Dim}(A\cap B)$$

$$= 2 - 1 = 1$$

$$\therefore \operatorname{Dim}\left(\frac{A+B}{A}\right) = \operatorname{Dim}\left(\frac{B}{A\cap B}\right)$$

$$\operatorname{Dim}(A+B) = \operatorname{Dim} A + \operatorname{Dim} B - \operatorname{Dim}(A\cap B)$$

$$= 2 + 2 - 1$$

$$= \underline{3}.$$

$$\operatorname{Dim}\left(\frac{A+B}{A}\right) = \operatorname{Dim}(A+B) - \operatorname{Dim} A$$

$$= 3 - 2$$

$$= \underline{1}.$$