KGS IAS

Maths Optional

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HW

$$y_2 - ay_1 = ay_1 - a^2y - b^2y$$

$$y_2 - 2ay_1 + (a^2+b^2)y = 0$$

$\rightarrow$ $y = e^{ax}(c_1 \sin bx + c_2 \cos bx)$

$$y_1 = ae^{ax}\underbrace{(c_1 \sin bx + c_2 \cos bx)}$$
$$+ e^{ax}(c_1 \cdot b \cdot \cos bx - c_2 \cdot b \sin bx)$$

$$y_1 - ay = be^{ax}(c_1 \cos bx - c_2 \sin bx)$$

$$y_2 - ay_1 = abe^{ax}(c_1 \cos bx - c_2 \sin bx)$$
$$+ be^{ax}(-c_1 b \sin bx - c_2 b \cos bx)$$

$$y_2 - ay_1 = a(y_1 - ay) - b^2 y$$

Variable separable

$$\frac{dy}{dx} = f(x, y)$$

$$\frac{dy}{dx} = \frac{f_1(x)}{f_2(y)}$$

$$\Rightarrow f_2(y)\,dy = f_1(x)\,dx$$

$$\int f_2(y)\,dy = \int f_1(x)\,dx + C.$$

Equation of 1st order and 1st degree

$$\frac{dy}{dx} = f(x, y)$$

or $M\,dx + N\,dy = 0$

(I) variable separable.

(II) Homogeneous D.E.

(III) Non-Homo. D.E.

(IV) Exact D.E.

(V) Linear D.E.

$$\int e^{-y}\,dy = \int e^{x}\,dx + \int x^2\,dx$$

$$\Rightarrow \frac{e^{-y}}{-1} = e^{x} + \frac{x^3}{3} + \text{const}$$

$$\Rightarrow -e^{-y} = e^{x} + \frac{x^3}{3} + \text{cont}$$

$$\Rightarrow e^{x} + e^{-y} + \frac{x^3}{3} = \text{const}$$

$$\Rightarrow 3\left(e^{x} + e^{-y}\right) + x^3 = C.$$

<u>prob:</u>

$$\frac{dy}{dx} = e^{x+y} + x^2 e^{y}$$

$$\Rightarrow \frac{dy}{dx} = e^{y}\left(e^{x} + x^2\right)$$

$$\Rightarrow \frac{dy}{e^{y}} = \left(e^{x} + x^2\right)dx$$

Integrate,

Int:

$$\log x - x = \log y + \text{const}$$

$$\Rightarrow \log x - \log y - x = \text{const}$$

$$\Rightarrow \log \frac{x}{y} = x + c$$

$$\Rightarrow \frac{x}{y} = e^{x+c}$$

$$\Rightarrow \frac{x}{y} = e^{x} \cdot e^{c}$$

$$\Rightarrow \frac{x}{y} = e^{x} \cdot c_1 \quad [c_1 = e^c]$$

$$\Rightarrow x = y e^{x} \cdot c_1$$

Prob: $y\,dx - x\,dy = xy\,dx$

$$\Rightarrow y\,dx - xy\,dx = x\,dy$$

$$\Rightarrow y(1-x)\,dx = x\,dy$$

$$\Rightarrow \frac{(1-x)}{x}\,dx = \frac{1}{y}\,dy$$

$$\Rightarrow \left(\frac{1}{x} - 1\right)dx = \frac{1}{y}\,dy$$

$$\left(\frac{1}{x} - x^{-2}\right)dx = \frac{dy}{y}$$

Integrate,

$$\log x - \frac{x^{-2+1}}{-2+1} = \log y + \text{const.}$$

$$\Rightarrow \log x - \log y + \frac{1}{x} = \text{const.}$$

$$\Rightarrow \log\frac{x}{y} + \frac{1}{x} = C.$$

prob: $\left(y - x\frac{dy}{dx}\right)\cdot x = y$

$$\Rightarrow xy - x^2\frac{dy}{dx} = y$$

$$\Rightarrow xy - y = x^2\frac{dy}{dx}$$

$$\Rightarrow y(x-1) = x^2\frac{dy}{dx}$$

$$\Rightarrow \frac{(x-1)}{x^2}dx = \frac{dy}{y}$$

$\left(\frac{1}{x}-1\right)dx+\left(\frac{1}{y}+1\right)dy=0$

Int.

$\log x - x + \log y + y = C$

$\Rightarrow \log(x \cdot y) - x + y = c$

At $x=1$, $y=1$

$\log 1 - \cancel{1} + \cancel{1} = C$

$c=0$

$\underline{\log(xy) - x + y = 0}$

Prob: Find the equation of the curve passing through the point $(1,1)$, whose D.E. is

$(y - yx)\,dx + (x + xy)\,dy = 0$

Solⁿ $y(1-x)\,dx + x(1+y)\,dy = 0$

$\Rightarrow \frac{(1-x)}{x}dx + \frac{(1+y)}{y}dy = 0$

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$$

Equations reducible to the form in which variables can be separated.

$$\rightarrow \quad \frac{dy}{dx} = f\left(\underline{ax + by + c}\right)$$

Let $z = ax + by + c$

$$\frac{dz}{dx} = a + b\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

$$\frac{dz}{dx} - 3 = z^2$$

$$\Rightarrow \frac{dz}{dx} = z^2 + 3$$

$$\Rightarrow \frac{dz}{z^2+3} = dx$$

Int.

$$\int \frac{dz}{(\sqrt{3})^2 + z^2} = \int dx$$

$$\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z}{\sqrt{3}}\right) = x + \text{const.}$$

prob: $\frac{dy}{dx} = (3x+y+4)^2.$

put $z = 3x+y+4$

$$\frac{dz}{dx} = 3 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 3$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

(1) $\Rightarrow$

$$\frac{dz}{dx} - 1 = \cos z + \sin z$$

$$\Rightarrow \frac{dz}{dx} = \underbrace{1 + \cos z} + \sin z$$

$$\Rightarrow \frac{dz}{1 + \cos z + \sin z} = dx$$

Integrating,

$$\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{3x + y + y}{\sqrt{3}}\right) = x + \text{Const.}$$

$$\tan^{-1}\left(\frac{3x + y + y}{\sqrt{3}}\right) = \sqrt{3}\, x + C$$

Proof: $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$ (1)

Put $z = x + y$

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \int \frac{\frac{1}{2}\sec^2 z/2}{1+\tan z/2}\,dz = x + \text{const}$$

$$\Rightarrow \log\left(1+\tan\frac{z}{2}\right) = x + \text{const}$$

$$\Rightarrow 1+\tan\frac{z}{2} = e^{x+\text{const}} = e^{x}\cdot c$$

$$\Rightarrow \boxed{1+\tan\left(\frac{x+y}{2}\right) = c\cdot e^{x}}$$

$$\int \frac{dz}{1+\cos z+\sin z} = x + \text{const.}$$

$$\Rightarrow \int \frac{dz}{2\cos^2\frac{z}{2} + 2\sin\frac{z}{2}\cdot\cos\frac{z}{2}} = x + \text{const}$$

$$\Rightarrow \int \frac{dz}{2\cos^2\frac{z}{2}\left[1+\tan\frac{z}{2}\right]} = x + \text{const}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

prob: $$\frac{dy}{dx} = \frac{x-y+3}{2(x-y)+5} \quad \text{—} (1)$$

Put $z = x - y$

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{dz}{dx}$$

Diff. eqn. of the form:

$$\frac{dy}{dx} = \frac{m(ax+by)+c}{ax+by+c_1}$$

or

$$\frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+c_1}$$

Put $z = ax + by$.

$$\frac{dz}{dx} = a + b\frac{dy}{dx}$$

$$\frac{z+2}{2z+5} = \frac{dz}{dx}$$

$$\Rightarrow \frac{2z+5}{z+2}\,dz = dx$$

integrate,

$$\int \frac{2z+5}{z+2}\,dz = \int dx$$

$$\Rightarrow \int \frac{2(z+2)+1}{z+2}\,dz = x + \text{const.}$$

$$① \Rightarrow$$

$$1 - \frac{dz}{dx} = \frac{z+3}{2z+5}$$

$$\Rightarrow 1 - \frac{z+3}{2z+5} = \frac{dz}{dx}$$

$$\Rightarrow \frac{2z+5-z-3}{2z+5} = \frac{dz}{dx}$$

Homogeneous D.E

Homogeneous fn:

$f(x,y)$ — homo. fn.
of degree 'n' when

$$f(kx, ky) = k^n f(x,y)$$

$$\Rightarrow 2\int dz + \int \frac{dz}{z+2} = x + \text{const.}$$

$$\Rightarrow 2z + \log(z+2) = x + \text{const.}$$

$$\Rightarrow 2(x-y) + \log(x-y+2) = x + \text{const}$$

$$\Rightarrow x - 2y + \log(x-y+2) = c.$$

Note: $f(x,y)$ — hom. fn. of degree 'n'.

$$f(x,y) = x^n \cdot F\left(\frac{y}{x}\right)$$

or

$$y^n \cdot G\left(\frac{x}{y}\right).$$

Ex. $f(x,y) = \dfrac{x^3 - y^3 + x^2 y}{x^2 + y^2}$

Ex: $f(x,y) = \dfrac{x^3 - y^3 + x^2 y}{x^2 + y^2}$

$$f(kx, ky) = \frac{k^3x^3 - k^3y^3 + k^3x^2y}{k^2x^2 + k^2y^2}$$

$$= k \cdot \frac{x^3 - y^3 + x^2 y}{x^2 + y^2}$$

$$= k^1 \cdot f(x,y)$$

Hom. fn. of degree '1'.

expressed as a fn. of x/y or y/x.

i.e. $f(x,y) = F\left(\frac{y}{x}\right)$

or

$G\left(\frac{x}{y}\right)$

$$= \frac{x^3\left[1-\left(\frac{y}{x}\right)^3+\frac{y}{x}\right]}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}$$

$$= x^1 \cdot F\left(\frac{y}{x}\right)$$

<u>Note</u>: If $f(x,y)$ is a homo. fn. of degree '0' then $f(x,y)$ can be

$$\frac{dy}{dx} = f(x, y)$$

put $y = vx$ or $x = vy$

suppose $y = vx$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

or

Suppose $x = vy$

$$\frac{dx}{dy} = v + y \cdot \frac{dv}{dy}$$

Homogeneous D.E.

$$\frac{dy}{dx} = \underbrace{f(x, y)}$$

homo. fn. of degree '0'.

$$\frac{f_1(x, y)}{f_2(x, y)}$$

f_1, f_2 – homo. of same degree.

$$(1) \Rightarrow$$

$$v + x\frac{dv}{dx} = \frac{v \cdot x^3}{x^3 + v^3 \cdot x^3}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$= \frac{\cancel{v} - \cancel{v} - v^4}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$$

Prob: $x^2 y\,dx - (x^3+y^3)\,dy = 0$

$$\Rightarrow x^2 y\,dx = (x^3+y^3)\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 y}{x^3+y^3} \quad \text{— (1)}$$

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

$$\Rightarrow -\frac{1}{3v^3} + \log v\cdot x = \text{const}$$

$$\Rightarrow -\frac{1}{3\left(\frac{y}{x}\right)^3} + \log\left(\frac{y}{x}\times x\right) = \text{const}$$

$$\Rightarrow -\frac{x^3}{3y^3} + \log y = \text{const} = C$$

$$\Rightarrow \log y = C + \frac{x^3}{3y^3}$$

$$\Rightarrow y = e^{\frac{x^3}{3y^3}}\cdot e^{C}$$

$$= C_1\cdot e^{\frac{1}{3}\left(\frac{x}{y}\right)^3}, \quad C_1 = e^{C}$$

$$\frac{1+v^3}{-v^4}\,dv = \frac{dx}{x}$$

$$\left(\frac{1+v^3}{v^4}\right)dv = -\frac{dx}{x}$$

$$\left(v^{-4} + \frac{1}{v}\right)dv = -\frac{dx}{x}$$

integrate

$$\frac{v^{-4+1}}{-4+1} + \log v = -\log x + \text{const}$$

HW prob: $\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$

Ans: $-e^{-y/x} = \log x + c$