KGS IAS

Maths Optional

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It is denoted by
$\{S\}$.

<u>Ex</u>: $S = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$

$L(S) = \{(a,b,c) : a,b,c \in \mathbb{R}\}$
$= \mathbb{R}^3$

$\mathbb{R}^3 = \{S\}$

Smallest subspace containing any subset of $V(F)$.

Let $V(F)$ be a vector space and S be any subset of V. If U is a subspace of V containing S and is contained in every subspace of V containing S, then U is called the smallest subspace containing S.

(i.e., $L(S)$ is a subspace of $V(F)$ generated by S. i.e., $\underline{L(S) = \{S\}}$.)

<u>proof</u>: Let $u \in S$

$u = 1 \cdot u$

a linear comb. of the elements of S.

$\Rightarrow u \in L(S)$

$\therefore \underline{S \subseteq L(S)}$.

<u>Note</u>: The smallest subspace containing S is called the subspace generated by S, denoted by $\{S\}$.

<u>Th</u>: If $V(F)$ is a vector space, $S \subseteq V$ then the linear span of S is the smallest subspace of $V(F)$ containing S.

Claim: $L(S)$ is a subspace of $V(F)$.

Let $\alpha, \beta \in F$

$u, v \in L(S)$

$\therefore u = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$

$a_i \in F$

$u_i \in S$

$v = b_1 v_1 + b_2 v_2 + \cdots + b_m v_m$

$b_i \in F$

$v_i \in S$

$$\alpha u + \beta v = \alpha(a_1 u_1 + a_2 u_2 + \cdots + a_n u_n) + \beta(b_1 v_1 + b_2 v_2 + \cdots + b_m v_m)$$

$$= (\alpha a_1) u_1 + (\alpha a_2) u_2 + \cdots (\alpha a_n) u_n + (\beta b_1) v_1 + (\beta b_2) v_2 + \cdots + (\beta b_m) v_m$$

= linear comb. of elements of S.

$\therefore \alpha u + \beta v \in L(S)$

$\therefore L(S)$ is a subspace of V.

$S \subseteq T$

$\therefore v_i \in T$

$\therefore a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \in T$

(As T is a subspace)

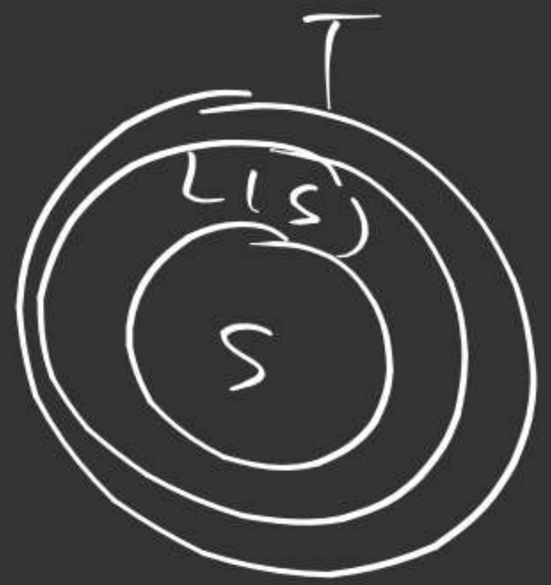
$\therefore v \in T$

$\Rightarrow L(S) \subseteq T.$

Now we prove that $L(S)$ is the smallest subspace containing S.

Let T be a subspace containing S.

i.e. $S \subseteq T$.



<u>To prove</u>: $L(S) \subseteq T$.

Let $v \in L(S)$

$\therefore v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$a_i \in F$

$v_i \in S$

Let $(a, b, c) \in V_3(F)$

$(a, b, c) = a(1, 0, 0)$
$+ b(0, 1, 0) + c(0, 0, 1)$

$\therefore (a, b, c) \in L(S)$

$\therefore V_3(F) \subseteq L(S)$

So $L(S) = V_3(F)$

Note: $L(S) = V$ (prove!)

$L(S) \subseteq V$ (obvious)

only we need to prove.

$$\underline{\underline{V \subseteq L(S)}}$$

Ex: Let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subseteq V_3(F)$

To prove: $L(S) = V_3(F)$

Linear sum of two sub-spaces

Let W_1 and W_2 be any two sub-spaces of $V(F)$ then the set $\{w_i + w_j : w_i \in W_1, w_j \in W_2\}$ is called the linear sum of W_1 & W_2 and is denoted by $W_1 + W_2$.

i.e., $W_1 + W_2 = \{w_i + w_j : w_i \in W_1, w_j \in W_2\}$.

Th: Let W_1 and W_2 are sub-space of $V(F)$, then the linear sum $W_1 + W_2$ is a sub-space of $V(F)$ and $W_1 + W_2 = L(W_1 \cup W_2)$.

$\therefore \alpha u + \beta v \in W_1 + W_2$

$\therefore W_1 + W_2$ is a subspace of $V(F)$.

<u>Claim</u>: $W_1 + W_2 = L(W_1 \cup W_2)$

$x \in W_1$, $0 \in W_2$

$\Rightarrow x + 0 = x \in W_1 + W_2$

$\therefore W_1 \subseteq W_1 + W_2$ —— (i)

$0 \in W_1$, $y \in W_2$

$\Rightarrow 0 + y = y \in W_1 + W_2$

$\therefore W_2 \subseteq W_1 + W_2$ —— (ii)

<u>proof</u>: Let $\alpha, \beta \in F$.

Let $u, v \in W_1 + W_2$

$\Rightarrow u = u_1 + u_2$ for some

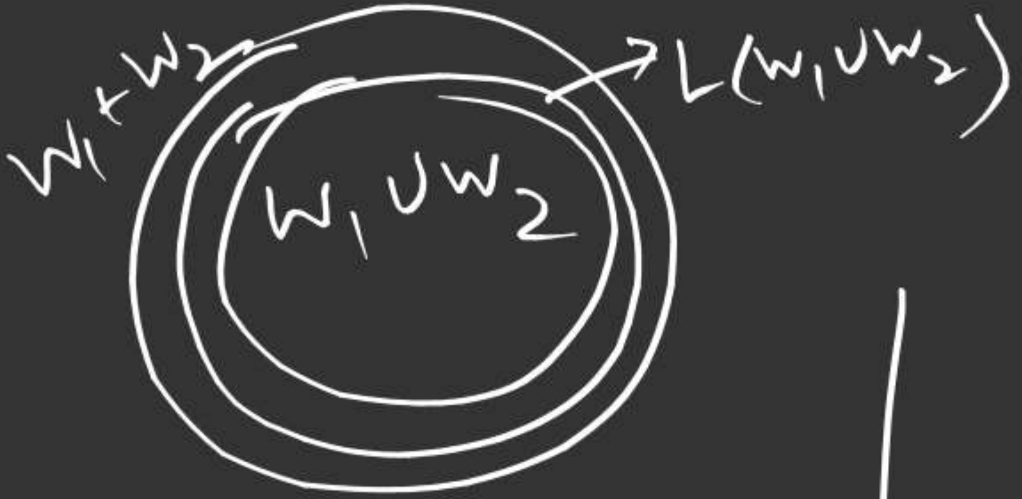
$v = v_1 + v_2$ $\quad u_1, v_1 \in W_1$

$u_2, v_2 \in W_2$

Now $\alpha u + \beta v$

$= \alpha(u_1 + u_2) + \beta(v_1 + v_2)$

$= \underbrace{(\alpha u_1 + \beta v_1)}_{\in W_1} + \underbrace{(\alpha u_2 + \beta v_2)}_{\in W_2}$



Let $v \in W_1 + W_2$

$\therefore v = v_1 + v_2$, for some $v_1 \in W_1$, $v_2 \in W_2$

$= 1 \cdot v_1 + 1 \cdot v_2$

L. comb. of elements of $W_1 \cup W_2$

$v_1, v_2 \in W_1 \cup W_2$

$\therefore v \in L(W_1 \cup W_2)$

From (i) and (ii)

$W_1 \cup W_2 \subseteq W_1 + W_2$

$W_1 + W_2$ is a subspace containing $W_1 \cup W_2$ and $L(W_1 \cup W_2)$ is the smallest sub-space containing $W_1 \cup W_2$

$\therefore L(W_1 \cup W_2) \subseteq W_1 + W_2$ — (iii) proof: (i) Let $u \in L(S)$

$\therefore u = a_1u_1 + a_2u_2 + \cdots + a_nu_n$

$a_i \in F$

$u_i \in S$

$\because S \subseteq T$

$\therefore u_i \in T$

$$u = \underbrace{a_1u_1 + a_2u_2 + \cdots + a_nu_n}_{\text{L. comb. of elements of } T}$$

$\therefore u \in L(T)$

$\Rightarrow L(S) \subseteq L(T)$

$\therefore W_1 + W_2 \subseteq L(W_1 \cup W_2)$ — (iv)

By (iii) and (iv)

$W_1 + W_2 = L(W_1 \cup W_2)$

Result: If S, T are subsets of $V(F)$ then

(i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

(ii) $L(S \cup T) = L(S) + L(T)$

(iii) $L(L(S)) = L(S)$

$S \subseteq L(S) \subseteq L(S)+L(T)$

$T \subseteq L(T) \subseteq L(S)+L(T)$

$\therefore S \cup T \subseteq L(S)+L(T)$

And $L(S \cup T)$ is the smallest subspace containing $S \cup T$.

$\therefore L(S \cup T) \subseteq L(S)+L(T)$ — (IV)

By (III) & (IV)

$\underline{L(S \cup T) = L(S)+L(T)}$

(II) $S \subseteq S \cup T$

$\Rightarrow L(S) \subseteq L(S \cup T)$ — (I)

By part (I)

$\therefore T \subseteq S \cup T$

$\Rightarrow L(T) \subseteq L(S \cup T)$ — (II)

$\because L(S)$ and $L(T)$ are subspace.

$\therefore$ (I) & (II) $\Rightarrow$

$L(S)+L(T) \subseteq L(S \cup T)$ — (III)

Prob: Is the vector $(3,-4,6)$ in the sub-space of $\mathbb{R}^3$ spanned by the vectors $(1,2,-1)$, $(2,2,1)$ and $(1,-2,3)$?

Soln: Let

$$(3,-4,6) = a(1,2,-1) + b(2,2,1) + c(1,-2,3)$$

(iii) $L(S) \subseteq L(L(S))$

Also $L(L(S)) \subseteq L(S)$

$\therefore L(L(S)) = L(S)$

$$a+2b+c=3$$

$$2a+2b-2c=-4$$

$$-a+b+3c=6$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 2 & -2 & -4 \\ -1 & 1 & 3 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -2 & -4 & -10 \\ 0 & 3 & 4 & 9 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 3 & 4 & 9 \end{array}\right]$$

$R_2 \to -\frac{1}{2} R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -2 & -6 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$a=2,\ b=-1,\ c=3$

Yes.

$r(A)=r(A:B)=3$

$=$ No. of var.

unique soln.

$-2c=-6 \Rightarrow c=3$

$b+2c=5$

$\Rightarrow b+6=5$

$\Rightarrow b=-1$

$a+2b+c=3$

$\Rightarrow a+(-2)+3=3$

$\Rightarrow a=2$.

Prob: If $v_1 = (1,2,1)$, $v_2 = (3,1,5)$ and $v_3 = (3,-4,7)$ are vectors in $\mathbb{R}^3$. prove that the sub-space spanned by $S = \{v_1, v_2\}$ and $T = \{v_1, v_2, v_3\}$ are the same.

Sol^n: Let $v_3 = a v_1 + b v_2$

$$(3,-4,7) = a(1,2,1) + b(3,1,5)$$

$$a + 3b = 3 \quad \text{—} \quad (\mathrm{I})$$

$$2a + b = -4 \quad \text{—} \quad (\mathrm{II})$$

$$a + 5b = 7 \quad \text{—} \quad (\mathrm{III})$$

$2 \times (\mathrm{I}) - (\mathrm{II})$

$$\cancel{2a} + 6b = 6$$

$$\cancel{2a} + b = -4$$

$$- \quad\quad - \quad\quad +$$

$$5b = 10 \Rightarrow b = 2$$

$(\mathrm{I}) \Rightarrow a + 6 = 3 \Rightarrow a = -3$

$a = -3,\ b = 2$ satisfy (III) also

$v \in L(S)$

$\therefore L(T) \subseteq L(S)$ — (VI)

$\therefore$ By (V) & (VI)

$L(S) = L(T)$

$\therefore v_3 = -3v_1 + 2v_2$ — (IV)

$\therefore S \subseteq T$

$\Rightarrow L(S) \subseteq L(T)$ — (V)

Let $v \in L(T)$

$\therefore v = a v_1 + b v_2 + c v_3$

$= a v_1 + b v_2 + c(-3v_1 + 2v_2)$

$= (a-3c)v_1 + (b+2c)v_2$ using (IV)

Prob: If v_1, v_2, v_3 are three vectors in a vector space $V(F)$ s.t. $v_1+v_2+v_3=0$ then show that $\{v_1, v_2\}$ spans the same sub-space as that by $\{v_2, v_3\}$.

Solⁿ: **T.P:** $L[\{v_1, v_2\}] = L[\{v_2, v_3\}]$

Let

$$v_1+v_2+v_3=0 \quad \text{——} \quad (1)$$

Let $v \in L[\{v_1, v_2\}]$

$$\Rightarrow v = av_1 + bv_2 \qquad a, b \in F$$

$$= a(-v_2-v_3) + bv_2 \qquad \left[\begin{array}{l}\text{from (1)} \\ v_1 = -v_2-v_3\end{array}\right]$$

$$= (b-a)v_2 - av_3$$

$$\Rightarrow v \in L[\{v_2, v_3\}]$$

$\therefore L[\{v_1, v_2\}] \subseteq L[\{v_2, v_3\}]$

Similarly, we can prove

$$L[\{v_2, v_3\}] \subseteq L[\{v_1, v_2\}]$$

$$\therefore L[\{v_1, v_2\}] = L[\{v_2, v_3\}]$$

Let V be a vector space over a field F. And $v_1, v_2, \dots, v_n$ be n vectors in V.

<u>Linear dependence</u>: The vectors $v_1, v_2, \dots, v_n$ are said to be linearly dependent (L.D) if there exists some scalars $a_1, a_2, \dots, a_n$ (not all zeros)

<u>s.t.</u>

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$$

<u>Ex</u>: $\underset{v_1}{(1,-1)}, \ \underset{v_2}{(4,-4)} \in \mathbb{R}^2$

$$v_1 - \frac{1}{4} v_2 = (0,0)$$

$$a_1 = 1, \ a_2 = -\frac{1}{4}$$

$$a_1 v_1 + a_2 v_2 = 0$$

v_1, v_2 — L.D.

Ex: $v_1 = (1,0)$
$v_2 = (0,1)$ $\in \mathbb{R}^2$

$a_1 v_1 + a_2 v_2 = (0,0)$

$\Rightarrow a_1(1,0) + a_2(0,1) = (0,0)$

$\Rightarrow (a_1, a_2) = (0,0)$

$\Rightarrow a_1 = 0,\ a_2 = 0$

v_1, v_2 are L.I.

Linear Independence: If the vectors $v_1, v_2, \ldots, v_n$ are not L.D. then they are L.I.

or

$v_1, v_2, \ldots v_n$ are said to be L.I. if

$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$

$\Rightarrow a_1 = a_2 = a_3 = \cdots = a_n = 0$

$a_i \in F$.