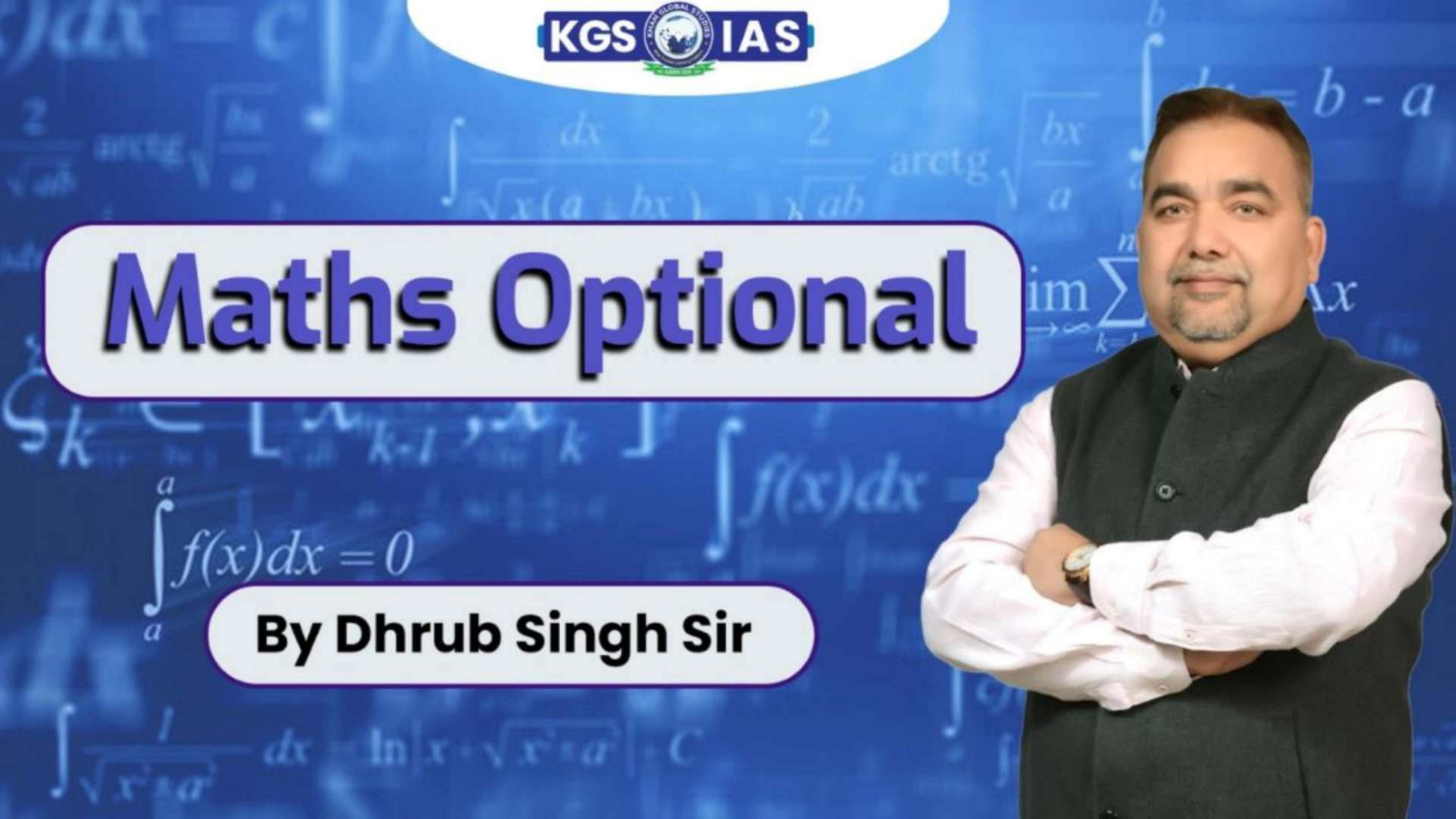
KGS IAS
Maths Optional
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Prob: If λ is a characteristic root of a matrix A, show that $k+\lambda$ is a characteristic root of the matrix $A+kI$.

Sol^n: Let λ is a ch. root of A and $X \neq 0$ be the corresponding eigen vector.

$$\therefore AX = \lambda X \quad \text{—— (1)}$$

$$(A+kI)X$$
$$= AX + kIX$$
$$= AX + kX$$
$$= \lambda X + kX \quad \text{using (1)}$$
$$= (\lambda + k)X$$

$X \neq 0$, X is the eigen vector corr. to the eigen value $\lambda + k$ of the mat. $A+kI$.

Prob: If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the characteristic values of an n-rowed square matrix A then show that $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ are the ch. values of kA.

Sol^n: Let $k \neq 0$ be a scalar.

$$|kA-(k\lambda)I| = |k(A-\lambda I)|$$

$$= k^n |A-\lambda I|$$

$\therefore |kA-(k\lambda)I| = 0$

iff $|A-\lambda I| = 0$

i.e. $k\lambda$ is a ch. root of kA iff λ is a ch. root of A.

i.e. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are ch. roots of A then $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ must be ch. roots of kA.

Now

$(A-KI)X$

$= AX - K(IX)$

$= AX - KX$

$= \lambda X - KX$

[using ①]

$= (\lambda - K)X$

$\therefore (A-KI)X = (\lambda-K)X$

$\therefore$ as $X \neq 0$, $\lambda - K$ is the

Prob: If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the ch. roots of the n-rowed square matrix A and K is a scalar, show that the ch. roots of $A - KI$ are

$\lambda_1 - K, \lambda_2 - K, \ldots \lambda_n - K.$

Soln: Let λ be a ch. root of A and X be the corr. eigen vector.

$$\therefore AX = \lambda X \quad \text{———} \quad ①$$

ch. root of $A - kI$.

and X is the corr. eigen vector.

i.e. if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Ch. root of A then

$\lambda_1 - k, \lambda_2 - k, \ldots \lambda_n - k$ are the ch. root of $A - kI$.

Prob: If the ch. roots of an n-rowed square matrix A are $\lambda_1, \lambda_2, \ldots \lambda_n$ then the ch. roots of A^2 are $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$.

Soln: Let λ be a ch. root of the matrix A and X be the corresponding eigen vector.

$$\therefore AX = \lambda X \quad \text{— (1)}$$

As $AX = \lambda X$

pre-multiplying both sides with A,

$$A(AX) = A(\lambda X)$$

$$\Rightarrow (AA)X = \lambda (AX)$$

$$\Rightarrow A^2 X = \lambda \cdot \lambda X \quad \text{using (1)}$$

$$\Rightarrow A^2 X = \lambda^2 X.$$

$\because X \neq 0$, $\therefore \lambda^2$ is the ch. root of A^2

and x is the corr. eigenvector.

i.e. if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the

ch. roots of A then

$\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$ are the

ch. roots of A^2.

prob: If the matrix A is non-singular then show that the eigen values of A^{-1} are reciprocals of the eigen values of A.

Soln: $\because$ A is non-singular.

$\therefore$ A^{-1} exists.

Let λ be a ch. root of A and X be the corr. eigen vector.

$\therefore \quad AX = \lambda X$

pre-multiplying both sides with A^{-1},

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$\Rightarrow (A^{-1}A)X = \lambda (A^{-1}X)$$

$$\Rightarrow IX = \lambda (A^{-1}X)$$

$$\Rightarrow X = \lambda (A^{-1}X)$$

$$\Rightarrow A^{-1}X = \left(\frac{1}{\lambda}\right) X$$

[$\because \lambda \neq 0$ as A is non-sing.]

$\therefore \frac{1}{\lambda}$ is the ch. root of A^{-1}.

And X is the corr. eigen vector.

Prob: Show that the two matrices A and $C^{-1}AC$ have the same ch. roots.

Soln: Let $B = C^{-1}AC$.

$$B - \lambda I = C^{-1}AC - \lambda I$$

$$= C^{-1}AC - C^{-1}(\lambda I)C$$

$$= C^{-1}(A - \lambda I)C$$

$$\Rightarrow |B - \lambda I| = |C^{-1}(A - \lambda I)\cdot C|$$

$$= |C^{-1}|\,|A - \lambda I|\cdot|C|$$

$$= \frac{1}{\cancel{|C|}}\cdot|A - \lambda I|\cdot\cancel{|C|}$$

$$= |A - \lambda I|$$

$$\therefore |B - \lambda I| = |A - \lambda I|$$

i.e. $|C^{-1}AC - \lambda I| = |A - \lambda I|$

$\therefore$ $C^{-1}AC$ and A have same ch. roots.

λ is an eigenvalue of A

iff $|A-\lambda I|=0$

i.e. iff $\overline{|A-\lambda I|}=0$

i.e. iff $|\overline{A-\lambda I}|=0$

i.e. iff $|(\overline{A-\lambda I})^T|=0$

i.e. iff $|(A-\lambda I)^\theta|=0$

i.e. iff $|A^\theta-\bar{\lambda} I^\theta|=0$

As ch. eqn. of $C^{-1}AC$ and A are same.

$\boxed{\text{Prob:}}$ If $\lambda \in \mathbb{C}$ (complex number set) is an eigen value of a square matrix A then prove that $\bar{\lambda}$ is an eigen value of A^θ and conversely.

or

Prove that eigen value of A^θ are the conjugate of eigen values of A.

iff

$\Longleftarrow$

i.e. iff $|A^\theta - \bar{\lambda} \cdot I| = 0$

i.e. iff $\bar{\lambda}$ is the eigen value of A^θ.

Prob: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

Sol^n:

$$|A - \lambda I| = 0$$

i.e.

$$\begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$\Rightarrow \lambda(\lambda - 5) + 2(\lambda - 5) = 0$$

$$\Rightarrow (\lambda + 2)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = -2, 5$$

$$3x_1+4x_2=0$$

$$\Rightarrow 3x_1=-4x_2$$

$$\Rightarrow x_1=-\frac{4}{3}x_2$$

let $x_2=k$.

$$\therefore x_1=-\frac{4}{3}k$$

$$X=\begin{bmatrix}-\frac{4}{3}k\\ k\end{bmatrix}=k\begin{bmatrix}-\frac{4}{3}\\ 1\end{bmatrix}$$

$$=k\cdot X_1$$

Corr. to $\lambda=-2$

Let $X=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ be the eigen vector.

$$(A-(-2)I)X=0$$

$$\begin{bmatrix}3&4\\3&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

i.e $$\begin{bmatrix}3&4\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$R_2\to R_2-R_1$

$\underline{\lambda = 5}$

$$(A - 5I)X = 0$$

$$\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

i.e. $\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\quad R_2 \to R_2 + \frac{3}{4}R_1$

i.e. $-4x_1 + 4x_2 = 0$

$\Rightarrow x_1 = x_2$

where $X_1 = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$

clearly, X_1 is L.I.

And X_1 is the eigen vector corr. to $\lambda = -2$.

Set of eigenvectors corr. to $\lambda = -2$ is $\{ kX_1 : k \text{ is a parameter} \}$.

Set of eigen vectors

Corr. to $\lambda = 5$ is

$\{ K \cdot X_2 : K$ is a parameter$\}$

Let $x_2 = K$

$\therefore x_1 = K$

$$X = \begin{bmatrix} K \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= K \cdot X_2$$

Where $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

X_2 is L.I.

$$\lambda^3 - 10\lambda^2 + 28\lambda - 24 = 0$$

$$(\lambda-2)\left[\lambda^2 - 8\lambda + 12\right] = 0$$

$$(\lambda-2)\left[\lambda^2 - 2\lambda - 6\lambda + 12\right] = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-6) = 0$$

$$\Rightarrow \lambda = 2, 2, 6$$

Prob: Find all the eigen values and the eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

Solⁿ:

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$x_1 + x_2 + x_3 = 0$

let $x_2 = k_1$

$x_3 = k_2$

$x_1 = -x_2 - x_3$

$= -k_1 - k_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -k_2 \\ 0 \\ k_2 \end{bmatrix}$$

Eigen vector corr. to $\lambda = 2$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

i.e. $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

X_1 and X_2 are eigen vectors corr. to $\lambda = 2$.

The set of eigen vectors corr. to $\lambda = 2$

is $\{ k_1 X_1 + k_2 X_2 :$

k_1 and k_2 are parameters, $\}$.

k_1 and k_2 can not be zero together.

$$X = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= k_1 X_1 + k_2 X_2$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

X_1 and X_2 are L.I.

Eigenvector corr. to $\lambda = 6$

$$(A - 6I)X = 0$$

$$\begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$