KGS IAS

Maths Optional

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Ex: If $V =$ the set of all 2×3 matrices with their element as rational numbers and $F = \mathbb{R}$. then $V(F)$ is not a vector space.

$\longrightarrow \alpha = \sqrt{2}$ $\qquad A \in V$

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\alpha A = \sqrt{2} A = \begin{bmatrix} -\sqrt{2} & 2\sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \notin V$$

Example: Let $V = \{(a_1, a_2, \dots a_n) : a_i \in F\}$

addition and scalar multiplication are defined as:

$$(a_1, a_2, \dots a_n) + (b_1, b_2, \dots b_n) = (a_1+b_1, a_2+b_2, \dots a_n+b_n)$$

$$\alpha(a_1, a_2, \dots a_n) = (\alpha a_1, \alpha a_2, \dots \alpha a_n)$$

$\alpha \in F$

$a_i's, b_i's \in F$

The V is a vector space over F.

→ Let $u = (a_1, a_2, \dots a_n) \in V$

① $v = (b_1, b_2, \dots b_n)$

$u + v = (a_1+b_1, a_2+b_2, \dots a_n+b_n) \in V$

$a_i, b_i \in F$

$\Rightarrow a_i + b_i \in F$

$\because a_i, b_i, c_i \in F$

$\therefore a_i + (b_i + c_i)$

$= (a_i + b_i) + c_i$

F is a field

$= [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n] + (c_1, c_2, \ldots c_n)$

$= (u + v) + w$

(11) $u = (a_1, a_2, \ldots a_n)$

$v = (b_1, b_2, \ldots b_n) \in V$

$w = (c_1, c_2, \ldots c_n)$

$u + (v + w)$

$= (a_1, a_2, \ldots a_n) + [b_1 + c_1, b_2 + c_2, \ldots b_n + c_n]$

$= [a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots a_n + (b_n + c_n)]$

$= [(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots (a_n + b_n) + c_n]$

$\|^{ly}$ $(-u)+u=0$

$-u$ — inverse of u

(V) $u+v=v+u$ (prove)

let $\alpha, \beta \in F$, $u=(a_1, - a_n)$
$v=(b_1, - b_n) \in V$

(VI) $\alpha(u+v)=\alpha(a_1+b_1, a_2+b_2 -- a_n+b_n)$

$=[\alpha(a_1+b_1), \alpha(a_2+b_2) -- \alpha(a_n+b_n)]$

(III) $0=(0, 0, -- 0) \in V$

$0 \in F$.

$0+u=u+0=u$.

(IV) For each $u \in V$

$\exists -u \in V$ s.t.

$u+(-u)=(a_1, a_2 -- a_n)$
$+(-a_1, -a_2 --- -a_n)$
$=(a_1-a_1, a_2-a_2, — a_n-a_n)$
$=0$

(VII) $(\alpha+\beta)u = \alpha u + \beta u$ (prove it!)

(VIII) $\alpha(\beta u) = (\alpha\beta)u$ "

(IX) $1.u = 1.(a_1, a_2 - a_n)$

$= (1.a_1, 1.a_2, - 1.a_n)$

$= (a_1, a_2, - a_n)$

$= u$

$\therefore$ V is a vector space over F.

$= [\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, - - \alpha a_n + \alpha b_n]$

$\alpha(a_i + b_i)$

$= \alpha a_i + \alpha b_i$

$\in F$.

$= (\alpha a_1, \alpha a_2, - \alpha a_n)$

$+ (\alpha b_1, \alpha b_2, - \alpha b_n)$

$= \alpha(a_1, a_2, - a_n) + \alpha(b_1, b_2 - b_n)$

$= \alpha u + \alpha v$

Notation : (1) $V_n(F)$ or F^n or $F^{(n)}$.

(11) $F = \mathbb{R}$

$\boxed{\mathbb{R}^n} = \{(a_1, a_2, \ldots a_n) : a_i \in \mathbb{R}\}$.

$a \cdot f(x) = \sum (a \cdot a_i) x^i$

$I(x) = 0$

$= 0 + 0 \cdot x + 0 \cdot x^2 + \cdots$

$f(x) \longrightarrow -f(x)$

$\downarrow \qquad \downarrow$

$\sum a_i x^i \qquad = \sum (-a_i) x^i$

$f(x) + (-f(x)) = \underline{I(x)}$

<u>Ex</u>: Let F be a field.

$$F[x] = \left\{ f(x) : f(x) = \sum_{i=0}^{\infty} a_i x^i, \ldots \ a_i \in F \right\}$$

<u>Int. Comp</u>: $f(x) = \sum a_i x^i$

$g(x) = \sum b_i x^i \quad \in F[x]$

$f(x) + g(x) = \sum (a_i + b_i) x^i$,

<u>Exter. Comp</u>: Let $a \in F$

Ex: Let F be a field.

Let $P_n = \{ f(x) : f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \; a_i \in F \}.$

Internal comp.

& External comp (previous ex.)

Ext. comp. Let $c \in F$.

$f \in V$

$$(c \cdot f)(x) = c \cdot f(x) \quad \forall x \in S.$$

To prove: V is a vector space over F.

Ex: Let F be any field. And S be any non-empty set.

Let $V = \{ f : f : S \to F$ be a fn. $\}$.

Int. comp.: $\forall f, g \in V$

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in S$$

(11) **Associativity:** Let $f, g, h \in V$

$$\{f+(g+h)\}(x)$$

$$= f(x) + (g+h)(x)$$

$$= f(x) + [g(x) + h(x)]$$

$$= [f(x) + g(x)] + h(x)$$

$$= (f+g)(x) + h(x)$$

$f(x), g(x), h(x) \in F$

Ass.

$$= [(f+g)+h](x)$$

(1) **Closure:** Let $f, g \in V$

$$(f+g)(x) = \underset{\in F}{f(x)} + \underset{\in F}{g(x)} \quad \forall x \in S$$

$$\therefore f+g : S \to F$$

$$\therefore f+g \in V.$$

$\forall x \in S$

$\therefore f+I=f.$

similarly, $I+f=f$

$\therefore f+I=I+f=f$

$\forall f \in V$

(IV) Existence of inverse:

Let $f \in V$

Define: $-f : S \to F$

as $(-f)(x) = -f(x) \; \forall x \in S$

$\forall x \in S$

$\therefore (f+g)+h = f+(g+h)$

(III) Existence of identity

$I : S \to F$

as

$I(x) = 0 \quad \forall x \in S$

$\in F$

$$(f+I)(x) = f(x) + I(x)$$
$$= f(x) + 0$$
$$[\because f(x), 0 \in F] = f(x)$$

Similarly,

$$(-f)+f = I$$

(v) Let $f, g \in V$

$$(f+g)(x) = f(x)+g(x)$$

$$= g(x)+f(x)$$

$$= (g+f)(x) \quad \forall x \in S$$

$f(x), g(x) \in F$

Comm.

$$\Rightarrow f+g = g+f.$$

Now

$$[f+(-f)](x)$$

$$= f(x)+(-f)(x)$$

$$= \underset{\in F}{f(x)} - \underset{\in F}{f(x)}$$

$$= 0 \in F$$

$$= I(x) \qquad \forall x \in S$$

$$\therefore f+(-f) = I$$

$\therefore a(f+g) = af + ag$

VII

VIII

IX

prove it

Let $a, b \in F$, $f, g \in V$

VI $[a \cdot (f+g)](x)$

$= a(f+g)(x)$

$= a[f(x) + g(x)]$

$= a \cdot f(x) + a \cdot g(x)$

$f(x), g(x) \in F$

Dist.

$= (af)(x) + (ag)(x)$

$= (af + ag)(x)$

$\forall x \in S$

Ext. comp. Let $\alpha \in \mathbb{R}$

$$(\alpha \cdot f)(x) = \alpha \cdot f(x)$$

$\forall x \in [0,1]$

Then V is a vector space over $\mathbb{R}$.

Ex: Let V be the set of all real valued continuous function defined on the closed interval $[0,1]$ i.e.

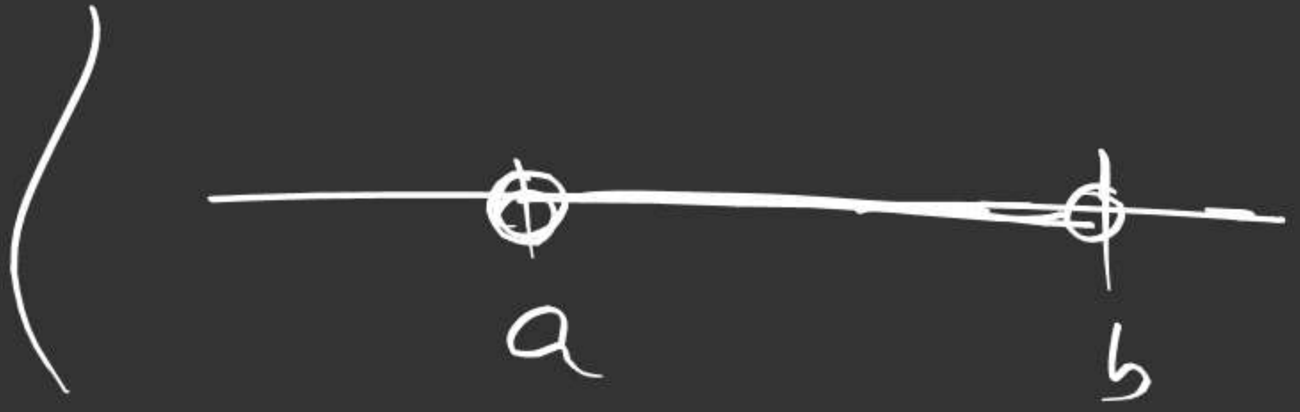
$$V = \{ f : f : [0,1] \to \mathbb{R},\ f \text{ is cts.} \}.$$

And $\mathbb{R}$ be the field of real number. For any $f, g \in V$

Int. comp: $(f+g)(x) = f(x) + g(x),\ \forall x \in [0,1]$

closed interval

$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$



$[a,b] = (a,b) \cup \{a,b\}$

$[a,b) = a \le x < b$

open interval

$]a,b[$ or (a,b)

$= \{x \in \mathbb{R} : a < x < b\}$

$]0,2[= \{x \in \mathbb{R} : 0 < x < 2\}$

prob: let $V = \{(x, y) : x, y \in \mathbb{R}\}$.

let $F = \mathbb{R}$

Examine in each of the following cases, whether V is a vector space over the field of real number or not?

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$c(x, y) = (cx, y)$

(ii) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 0)$

$c(x, y) = (cx, 0)$

$\alpha(x, y) + \beta(x, y)$

$= (\alpha x, y) + (\beta x, y)$

$= (\alpha x + \beta x, \quad 2y)$

$\therefore (\alpha+\beta)(x, y) \neq \alpha(x, y) + \beta(x, y)$

(iii) $(x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$c(x, y) = (c^2 x, c^2 y)$

Sol^n (i) V is not a v. space

$(\alpha+\beta)(x, y)$

$= [(\alpha+\beta)x, y)$

$= (\alpha x + \beta x, y)$

(III) Not a vector space.

$$(\alpha+\beta)(x, y)$$

$$=\left[(\alpha+\beta)^2 x, (\alpha+\beta)^2 y\right]$$

$$\alpha(x, y)+\beta(x, y)$$

$$=(\alpha^2 x, \alpha^2 y)+(\beta^2 x, \beta^2 y)$$

$$=\left[(\alpha^2+\beta^2) x, (\alpha^2+\beta^2) y\right]$$

(II) Not a vector space

$$1\cdot(x, y)=(1\cdot x, 0)$$

$$=(x, 0)$$

$$\neq (x, y)$$

(VI) $a\cdot v=\bar{0}$

$\Rightarrow a=0$ or $v=\bar{0}$.

proof:

(I) $\bar{0}+\bar{0}=\bar{0} \in V$

$\Rightarrow a\cdot(\bar{0}+\bar{0})=a\cdot\bar{0}$

$\Rightarrow a\cdot\bar{0}+a\cdot\bar{0}=\underset{\in V}{a\cdot\bar{0}}+\bar{0}$

$\Rightarrow a\cdot\bar{0}=\bar{0}$

[by left cancellation law in V]

Result: Let V(F) be a vector space and $\bar{0}$ be the zero vector of V. Then

(I) $a\cdot\bar{0}=\bar{0}$ $\forall a\in F$.

(II) $0\cdot v=\bar{0}$ $\forall v\in V$

(III) $a(-v)=-(av)$, $\forall a\in F$, $\forall v\in V$.

(IV) $(-a)v=-(av)$ "

(V) $a(v-u)=av-au$. $\forall a\in F$ $\forall u, v\in V$.

(iii) $a[v+(-v)]$

$= a\cdot\bar{0} = \bar{0}$

[By (i)]

$\Rightarrow av + a(-v) = \bar{0}$

$\Rightarrow$ Inverse of av is $a(-v)$

$\therefore -av = a(-v)$

(ii) $0+0=0$ $\quad 0 \in F$

$\Rightarrow (0+0)\cdot v = 0\cdot v$ $\quad v \in V$

$\Rightarrow \underbrace{0\cdot v}_{} + 0\cdot v = \underbrace{0\cdot v}_{\in V} + \underbrace{\bar{0}}_{\in V}$

$\Rightarrow 0\cdot v = \bar{0}$ [by left Cancellation law $\in V$]

(V) $a(v-u)$

$= a[v+(-u)]$

$= av + a(-u)$

$= av - au$

using (II)

(VI) Let $a \neq 0 \in F$

$\therefore a^{-1}$ exists.

(IV) $[a+(-a)]\cdot v = 0\cdot v = \bar{0}$ $\in F$ [By (II)]

$\Rightarrow av + (-a)\cdot v = \bar{0}$

$\therefore$ Inverse of av is $-a(v)$

So $-(av) = (-a)\cdot v$

Now let $v \neq \bar{0}$

Let if possible, $a \neq 0$

$\therefore a^{-1}$ exists.

$$a v = \bar{0}$$

$$\Rightarrow a^{-1}(a v) = a^{-1} \cdot \bar{0} = \bar{0}$$

$$\Rightarrow (a^{-1} a) \cdot v = \bar{0}$$

$$\Rightarrow 1 \cdot v = \bar{0}$$

$$\Rightarrow v = \bar{0}$$

a contradiction.

$\therefore a = 0.$

$\because a v = \bar{0}$

$$\Rightarrow a^{-1}(a v) = a^{-1} \cdot \bar{0}$$

$$\Rightarrow (a^{-1} \cdot a) v = \bar{0}$$

$$\Rightarrow 1 \cdot v = \bar{0}$$

$$\Rightarrow v = \bar{0}.$$

$\therefore a v = \bar{0}$

$\Rightarrow$ either $a = 0$ or $v = \bar{0}$

Sub-space: Let $V(F)$ be a vector space and $W \subseteq V$. If W is a vector space w.r.t. the internal composition and external composition of V then W is called a subspace of V.