Maths Optional

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Unitarily Similar matrices

Let A and B be two matrices of order 'n'. Then B is said to be unitarily similar to A if $\exists$ a unitary matrix P s.t.

$$P^{-1} A P = B$$

Results: (1) If A and B are unitarily similar then A and B are also similar.

of a Hermitian matrix are orthogonal.

(5) If λ (eigen value) occurs exactly 'p' times as an eigen value of a Hermitian matrix A then A has 'p' but not

(2) Every Hermitian matrix is unitarily similar to a Diagonal matrix.

(3) An $n \times n$ Hermitian matrix has 'n' mutually orthogonal complex eigen vectors.

(4) Any two eigen vectors corresponding to two distinct eigen values

$|A-\lambda I|=0$

$\Rightarrow \lambda^2-9=0$

$\Rightarrow \lambda^2=9$

$\Rightarrow \lambda=3,-3$

more than p mutually orthogonal eigen vectors corr. to λ.

Prob: Determine the diagonal matrix unitarily similar to the Hermitian matrix

$$A=\begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}.$$ Also find the transformation matrix.

$\therefore$ Rank of coeff. matrix $= 1$

No. of L.I. soln $= 2 - 1 = 1$

$$-x_1 + (1-2i)x_2 = 0$$

Let $x_2 = 1$

$\therefore x_1 = 1 - 2i$

$X_1 = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$ is the

Eigen vector corr. to $\lambda = 3$ is given by

$$(A - 3I)X = 0$$

$$\begin{bmatrix} -1 & 1-2i \\ 1+2i & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1-2i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$R_2 \to R_2 + (1+2i)R_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{5}(1-2i) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$R_2 \to R_2 - (1+2i)R_1$

$\therefore$ Rank of coeff. matrix $= 1$

$\therefore$ No. of LI sol^n. $= 2 - 1 = 1$

eigen vector corr. to $\lambda = 3$

Eigen vector corr. to $\lambda = -3$ is given by

$$(A + 3I)X = 0$$

$$\begin{bmatrix} 5 & 1-2i \\ 1+2i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{5}(1-2i) \\ 1+2i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$R_1 \to \frac{1}{5} R_1$

$X_2 = \begin{bmatrix} 1-2i \\ -5 \end{bmatrix}$ is the eigenvector corr. to $\lambda = -3$.

X_1 and X_2 are orthogonal ($\because$ eigenvalues are distinct)

$x_1 + \frac{1}{5}(1-2i)x_2 = 0$

Let $x_1 = 1-2i$

$\therefore \underline{(1-2i)}\left[1 + \frac{1}{5}x_2\right] = 0$

$\Rightarrow 1 + \frac{1}{5}x_2 = 0$

$\Rightarrow x_2 = -5$

$$\frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1-2i \\ -5 \end{bmatrix}$$

$$\|X_2\| = \sqrt{X_2^{\theta} \cdot X_2}$$

$$= \sqrt{(1+2i)(1-2i)+25}$$

$$= \sqrt{30}$$

Normalize X_1 and X_2

$$X_1 = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

$$\|X_1\| = \sqrt{X_1^{\theta} \cdot X_1}$$

$$= \sqrt{\overline{(1-2i)} \cdot (1-2i) + 1 \cdot 1}$$

$$= \sqrt{(1+2i)(1-2i)+1}$$

$$= \sqrt{5+1} = \sqrt{6}.$$

$$\hat{x}_2 = \frac{x_2}{\|x_2\|}$$

$$= \frac{1}{\sqrt{30}} \begin{bmatrix} 1-2i \\ -5 \end{bmatrix}$$

$$P = \begin{bmatrix} \hat{x}_1 , \hat{x}_2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

③ $A = I - 2uu^T$

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n \times 1}$$

where $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$

① $A^T = (I - 2uu^T)^T$

$= (I)^T - (2uu^T)^T$

$= I - 2\left[(u^T)^T \cdot u^T\right]$

$= I - 2uu^T = A$

$= I - \cancel{4uu^T} + \cancel{4uu^T}$

$= I$

$u^T \cdot u = \begin{bmatrix} a_1 & a_2 & - & a_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ | \\ a_n \end{bmatrix}$

$= \begin{bmatrix} a_1^2 + a_2^2 + \dots + a_n^2 \end{bmatrix}_{1\times 1}$

$= a_1^2 + a_2^2 + \dots + a_n^2$

$= 1$

11

$A^T \cdot A$

$= (I - 2uu^T)^T \cdot (I - 2uu^T)$

$= \left(I^T - (2uu^T)^T\right) \cdot (I - 2uu^T)$

$= \left(I - 2(u^T)^T u^T\right)(I - 2uu^T)$

$= (I - 2uu^T)(I - 2uu^T)$

$= I - 2uu^T - 2uu^T + 4u\underline{u^T u}u^T$

$$uu^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}_{1\times n}$$

$$= \begin{bmatrix} u_{11} & & & & \\ & u_{22} & & & \\ & & u_{33} & & \\ & & & \ddots & \\ & & & & u_{nn} \end{bmatrix}_{n\times n}$$

$u_{11} = a_1^2$ $\quad u_{nn} = a_n^2$

$u_{22} = a_2^2$

$\cdots \quad \cdots$

$tr(uu^T) = a_1^2 + a_2^2 + \cdots + a_n^2$

$= 1$

(iii) $A = I - 2uu^T$

$tr(A) = tr(I - 2uu^T)$

$= tr(I) - 2\,tr(uu^T)$

$= n - 2 \times 1$

$= n - 2$

$C_{3\times3}$

And $r(C) \le 2$

$\therefore |C| = 0$

$\Rightarrow C$ is singular.

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(8) $A_{3\times2}$ $\qquad B_{2\times3}$

To prove: $C = AB$ is singular.

$r(A) \le 2$, $r(B) \le 2$

$r(C) = r(AB)$

$\le \min(r(A), r(B))$

$\Rightarrow r(C) \le 2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & a-2 & 1 & 2 \\ 0 & 0 & \frac{a^2-4a-5}{a-2} & b-1-\frac{18}{a-2} \end{array}\right] \quad 3\times 4$$

$$R_3 \to R_3 - \frac{9}{a-2} R_2$$

$$r(A:B) = r(A) = 3$$

if $\dfrac{a^2-4a-5}{a-2} \neq 0$

$$a^2-4a-5=0$$

9

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 1 & a & 3 & 3 \\ 1 & 11 & a & b \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & a-2 & 1 & 2 \\ 0 & 9 & a-2 & b-1 \end{array}\right]$$

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 - R_1$$

$$a^2 - 5a + a - 5 = 0$$

$$\Rightarrow a(a-5) + 1(a-5) = 0$$

$$\Rightarrow (a+1)(a-5) = 0$$

$$\Rightarrow \underline{\underline{a = -1, 5}}$$

$\forall a \in \mathbb{R}$ except -1 & 5

12 $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$A^{12} = (A^3)^4$

$= 0$

$A^{14} = A^{12} \cdot A^2$

$= 0$

$\lambda^3 = 0$

$A^3 = 0$

$A^{14} + 3A - 2I$

$= 3A - 2I$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$y = 0$$
$$x = 0$$
$$z = k$$

$$X = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$