KGS IAS

Maths Optional

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$\Rightarrow x\cos\theta - y\sin\theta = 0$ —— (i)

$x\sin\theta + y\cos\theta = 0$ —— (ii)

$z = 0$ —— (iii)

Squaring (i) & (ii) and then adding, we have

$(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 = 0$

Prob: A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ is difened by $T(x, y, z) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ z)$

Show that T is non-singular.

Solⁿ: Let $u = (x, y, z) \in N(T)$

$\therefore T(x, y, z) = (0, 0, 0)$

$\Rightarrow (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ z) = (0, 0, 0)$

Result: Show that a linear transformation $T: U \to V$ over the field F is non-singular iff T is 1-1.

proof: Suppose T is non-singular

$\therefore N(T) = \{0\}$.

To prove: T is 1-1

$$x^2\cos^2\theta + y^2\sin^2\theta - 2xy\sin\theta\cdot\cos\theta + x^2\sin^2\theta + y^2\cos^2\theta + 2xy\sin\theta\cdot\cos\theta = 0$$

$$\Rightarrow x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) = 0$$

$$\Rightarrow x^2 + y^2 = 0$$

$$\Rightarrow x = 0,\ y = 0$$

$\therefore v = (0, 0, 0)$

$\therefore N(T) = \{0\} \Rightarrow T$ is non-singular

Conversely, suppose that T is 1-1.

To prove: T is non-singular.

Let $v \in N(T)$

$\Rightarrow T(v) = 0$

$\Rightarrow T(v) = T(0)$

$\Rightarrow v = 0$ [$\because T$ is 1-1]

$\therefore N(T) = \{0\}$

$\therefore T$ is non-singular.

Let $T(x) = T(y)$, $x, y \in U$

$\Rightarrow T(x) - T(y) = 0$

$\Rightarrow T(x-y) = 0$ [$\because T$ is L.T.]

$\Rightarrow x - y \in N(T) = \{0\}$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$.

$\therefore T$ is 1-1

$\Rightarrow$ Dim $N(T) = 0$

[$\because$ Dim U = Dim R(T)]

$\Rightarrow N(T) = \{0\}$.

$\therefore$ T is non-singular.

Conversely: Suppose T is non-singular.

To prove: Dim U = Dim R(T)

$\because$ T is non-singular.

$\therefore N(T) = \{0\}$.

Result: Let $T: U(F) \to V(F)$ be a LT. U is a FDVS. prove that U and Range T have the same dimension iff T is non-singular.

proof: Suppose Dim U = Dim R(T)

To prove: T is non-singular

By Rank-Nullity th.

$$\text{Dim } R(T) + \text{Dim } N(T) = \text{Dim } U$$

Result: Let U and V are FDVS of the same dimension over the field F. A linear mapping $T: U \to V$ is 1-1 iff it is onto.

Proof: T is 1-1 $\iff$ T is non-sing.

$\iff N(T) = \{0\}$

$\iff \text{Dim } N(T) = 0$

$\therefore \text{Dim } N(T) = 0$

By Rank-Nullity th,

$\text{Dim } R(T) + \text{Dim } N(T) = \text{Dim } U.$

$\implies \text{Dim } R(T) + 0 = \text{Dim } U$

$\implies \text{Dim } R(T) = \text{Dim } U$

Inverse function: Let $T: U \to V$ be a one-one onto mapping. Then the mapping $T^{-1}: V \to U$ defined by

$$T^{-1}(v) = u \Leftrightarrow T(u) = v,$$

$u \in U$, $v \in V$ is called the inverse mapping of T.

Note: T^{-1} is also 1-1 onto.

(iff)

$\Leftrightarrow \operatorname{Dim} U = \operatorname{Dim} R(T)$

$\Leftrightarrow \operatorname{Dim} V = \operatorname{Dim} R(T)$ [By Rank-nullity th.]

$\Leftrightarrow V = R(T)$ [$\because \operatorname{Dim} U = \operatorname{Dim} V$]

$\Leftrightarrow T$ is onto.

Proof: $\because T$ is 1-1-onto.

$\therefore T^{-1}$ exists.

and so T is invertible.

To prove: T^{-1} is a L.T.

$$T^{-1} : V \to U$$

Let $a, b \in F$,

Let $v_1, v_2 \in V$

Result: Let $U(F)$ and $V(F)$ be two vector spaces and $T : U \to V$ be a one-one and onto linear transformation. Then T^{-1} is also a linear trans. and T is said to be invertible.

$\Rightarrow au_1 + bu_2 = T^{-1}(av_1 + bv_2)$

$\Rightarrow aT^{-1}(v_1) + b \cdot T^{-1}(v_2) = T^{-1}(av_1 + bv_2)$

i.e. $T^{-1}(av_1 + bv_2)$

$= a \cdot T^{-1}(v_1) + b \cdot T^{-1}(v_2)$

$\therefore T^{-1}$ is a L.T.

$\because T : U \to V$ and T is onto

$\therefore \exists\, u_1, u_2 \in U$

s.t

$T(u_1) = v_1 \Leftrightarrow T^{-1}(v_1) = u_1$

$T(u_2) = v_2 \Leftrightarrow T^{-1}(v_2) = u_2$

Now $T(au_1 + bu_2)$

$= aT(u_1) + bT(u_2)$

$= av_1 + bv_2$

Th: Let $U(F)$ and $V(F)$ be two finite dimensional vector spaces such that $\dim U = \dim V$. A linear transformation $T : U \to V$ is invertible iff T is non-singular.

proof: Suppose T is non-singular.

To prove: T is invertible.

i.e., **To prove:** T is 1-1 & onto.

T is 1-1

$$T(x) = T(y)$$

$$\Rightarrow T(x) - T(y) = 0$$

$$\Rightarrow T(x-y) = 0$$

$\text{Dim } R(T) + \underbrace{\text{Dim } N(T)}_{0} = \text{Dim } V$

$\Rightarrow \text{Dim } R(T) = \text{Dim } U$

$= \text{Dim } V$ (given)

$\Rightarrow \text{Dim } R(T) = \text{Dim } V$

$\Rightarrow R(T) = V$

$\Rightarrow$ T is onto

$\Rightarrow \underline{x - y} \in N(T) = \{0\}$.

$\Rightarrow x - y = 0$ [$\because$ T is non-singular]

$\Rightarrow x = y$

T is onto: $N(T) = \{0\}$.

$\text{Dim } N(T) = 0$

By Rank-Nullity th.

$\therefore N(T) = \{0\}$.

$\therefore$ T is non-singular.

Suppose T is invertible.

$\therefore T^{-1}$ exist

Means T is 1-1 & onto.

To prove: T is non-singular

Let $v \in N(T)$

$\Rightarrow T(v) = 0$

$\Rightarrow T(v) = T(0)$ [$\because$ T is L.T. $\therefore T(0) = 0$]

$\Rightarrow v = 0$ [$\because$ T is 1-1]

(III) Range $T = V$

(IV) If $\{v_1, v_2, \ldots, v_n\}$ is any basis of U then $\{T(v_1), T(v_2) \ldots T(v_n)\}$ is a basis of V.

$\longrightarrow$

$(I) \Rightarrow (II) \Rightarrow (III) \Rightarrow (IV) \Rightarrow (I)$

Result: Let $U(F)$ and $V(F)$ be two finite dimensional vector spaces such that $\dim U = \dim V$. If $T : U \to V$ is a linear transformation then followings are equivalent:

(I) T is invertible

(II) T is non-singular

$\because T(x, y, z) = (a, b, c)$

$\Rightarrow (2x, 4x-y, 2x+3y-z) = (a, b, c)$

$\Rightarrow \quad 2x = a \Rightarrow x = a/2$

$4x - y = b \Rightarrow y = 4x - b$
$= 2a - b$

$2x + 3y - z = c$

$\Rightarrow z = 2x + 3y - c$
$= a + 6a - 3b - c$
$= 7a - 3b - c$

Prob: If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is invertible operator defined by $T(x, y, z) = (2x, 4x-y, 2x+3y-z)$. Find T^{-1}.

$\rightarrow \quad T(x, y, z) = (a, b, c)$

$\Leftrightarrow (x, y, z) = T^{-1}(a, b, c)$

$T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$. Show that T is non-singular and find its inverse.

$\rightarrow$ Let $(x, y, z) \in V_3(\mathbb{R})$

$$T(x, y, z) = T(xe_1 + ye_2 + ze_3)$$
$$= xT(e_1) + yT(e_2) + zT(e_3)$$
$$= x(e_1 + e_2) + y(e_2 + e_3) + z(e_1 + e_2 + e_3)$$

$$\therefore T^{-1}(a, b, c) = \left(\frac{a}{2}, 2a - b, 7a - 3b - c\right)$$

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prob: The set $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(\mathbb{R})$, $T : V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ is a linear operator defined by

$\Rightarrow x + z = 0$

$x + y + z = 0 \Rightarrow y = 0$

$y + z = 0$

$\Rightarrow z = 0$

$\Rightarrow x = 0$

$\therefore v = (0, 0, 0)$

$\therefore N(T) = \{0\}.$

$= x(1, 1, 0) + y(0, 1, 1)$

$+ z(1, 1, 1)$

$T(x, y, z) = (x + z, x + y + z, y + z)$

T is non-singular.

Let $v = (x, y, z) \in N(T)$

$\therefore T(x, y, z) = (0, 0, 0)$

$\Rightarrow (x + z, x + y + z, y + z) = (0, 0, 0)$

(ii) − (i)

$$y = b - a$$

(ii) − (iii)

$$x = b - c$$

Now (ii)

$$b - c + \cancel{b} - a + z = \cancel{b}$$

$$\Longrightarrow z = a - b + c$$

$$\therefore T^{-1}(a, b, c) = (b - c,\ b - a,\ a - b + c)$$

Let $T(x, y, z) = (a, b, c)$

$$\Longleftrightarrow (x, y, z) = T^{-1}(a, b, c)$$

$$\because T(x, y, z) = (x + z,\ x + y + z,\ y + z)$$

$$\Longrightarrow (a, b, c) = (x + z,\ x + y + z,\ y + z)$$

$$\Longrightarrow x + z = a \quad \text{— (i)}$$

$$x + y + z = b \quad \text{— (ii)}$$

$$y + z = c \quad \text{— (iii)}$$

(iii) prove that

$$(T^2 - I)(T - 3I) = 0$$

→ (ii) $N(T) = \{0\}$.

$T^{-1}(a, b, c)$

$$= \left(\frac{a}{3}, \frac{a}{3} - b, -a + b + c\right)$$

prob: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $T(x, y, z) = (3x, x - y, 2x + y + z)$, $\forall (x, y, z) \in \mathbb{R}^3$

(i) Show that T is a linear operator on $\mathbb{R}^3$.

(ii) Is T invertible? If yes, find T^{-1}!

$$T^2(x,y,z)$$
$$= T(T(x,y,z))$$
$$= T(3x, x-y, 2x+y+z)$$
$$= (9x, 2x+y, 6x+x-y + 2x+y+z)$$
$$= (9x, 2x+y, 9x+z)$$

Let $(x,y,z) \in \mathbb{R}^3$

$$(T-3I)(x,y,z)$$
$$= T(x,y,z) - 3I(x,y,z)$$
$$= (3x, x-y, 2x+y+z) - 3(x,y,z)$$
$$= (0, x-4y, 2x+y-2z)$$

$\therefore (T^2 - I)(T - 3I)(x, y, z)$

$= O(x, y, z)$

$\forall (x, y, z) \in \mathbb{R}^3.$

$\Rightarrow (T^2 - I)(T - 3I) = O$

$(T^2 - I)(T - 3I)(x, y, z)$

$= (T^2 - I)[(T - 3I)(x, y, z))]$

$= (T^2 - I)[0, x - 4y, 2x + y - 2z]$

$= T^2(0, x - 4y, 2x + y - 2z)$

$- I(0, x - 4y, 2x + y - 2z)$

$= (0, x - 4y, 2x + y - 2z) - (0, x - 4y, 2x + y - 2z)$

$= (0, 0, 0) = O(x, y, z)$