KGS IAS

Maths Optional

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ODE

(I) Formation of D.E.

(II) D.E. of 1st order & 1st degree.

(III) D.E. of 1st order but not 1st degree

(IV) L.D.E. with constant coeff.

(V) L.D.E. of 2nd order with variable coeff.

(VI) orthogonal trajectories.

(VII) Laplace transform.

$\frac{dy}{dx} \to y'$

$y^{(1)}$ or y_1

$\frac{d^2y}{dx^2} \to y''$

$y^{(2)}$ or y_2

Diff. eqn.

$y = f(x)$

Dependend var.

x - independent var.

ordinary Diff. eqn (I) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + e^x$

Partial Diff. eqn. (II) $\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x \cdot \partial y} + x^2\frac{\partial^2 z}{\partial y^2} = 0$

Sol^n

- General sol^n.
- particular sol^n.
- Singular sol^n.

$\psi(x, y) = 0$

$$\frac{dy}{dx} = \frac{d\, c\cdot e^{3x}}{dx} = c\,\frac{d e^{3x}}{dx} = c\cdot e^{3x}\cdot 3 = 3y$$

$$\frac{dy}{dx} = 3y$$

Gen sol^n: $y = c\cdot e^{3x}$

c → arbitrary const.

particular sol^n

$y = 2e^{3x}$

Ex: $\frac{d^3y}{dx^3} = \left[x + \left(\frac{dy}{dx}\right)^3\right]^{2/3}$

order = 3.

Ex: $\log\left(\frac{dy}{dx}\right) = 2x - 3y$

order = 1

Ex: $\cos\left(\frac{d^2y}{dx^2}\right) = x^3$

Order = 2.

order of a diff. eqn.

Highest order derivative

↓

order

↓

order of D.E.

Ex: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

Order = 2

Ex: $x \cdot \left(\frac{d^2y}{dx^2}\right)^3 + 2\frac{dy}{dx} + y = e^x$

order = 2

Degree = 3

Degree of a diff. eqn.

Degree of highest order derivatives.

Derivatives in the D.E should be free from radicals, fractions etc.

Ex:

$$\left(\frac{d^3y}{dx^3}\right)^{4/3} + \sin x \cdot \frac{dy}{dx} + y = x$$

$$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^{4/3} = x - \sin x \frac{dy}{dx} - y$$

cubing both sides,

$$\left(\frac{d^3y}{dx^3}\right)^{4} = \left(x - \sin x \frac{dy}{dx} - y\right)^3$$

order = 3
Degree = 4

Ex:

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{5/2}$$

squaring both sides,

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^5$$

order = 2
Degree = 2

Ex: $x\frac{dy}{dx} = \log\left(3y - \frac{d^2y}{dx^2}\right)$

order = 2

Degree = not defined.

Ex: $y = \sin\left(\frac{dy}{dx}\right)$

order = 1

Degree = not defined.

$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - - \quad -$

$\sin\left(\frac{dy}{dx}\right) = \frac{dy}{dx} - \frac{1}{\lfloor 3}\left(\frac{dy}{dx}\right)^3 + \frac{1}{\lfloor 5}\left(\frac{dy}{dx}\right)^5$

(iii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 0$

Non-linear Diff. eqn

(i) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + xy = 0$

(ii) $\frac{\partial^2 z}{dx^2} = 2\left(\frac{\partial^3 z}{\partial y^3}\right)^3$

Linear Diff. eqn: A diff. eqn. is called linear when dependent variable and its derivatives occur in 1st degree, not multiplied together.

(i) $\frac{dy}{dx} - xy = e^x$

(ii) $\frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + xy = x^2 + 1$

Formation of a D.E.

prob: Find the diff. eqn. of $y = A\cdot e^{2x} + B\cdot e^{-3x}$, where A and B are arb. const.

Soln:

$$y = A\cdot e^{2x} + B\cdot e^{-3x} \quad \text{—— (1)}$$

$$\frac{dy}{dx} = 2Ae^{2x} - 3Be^{-3x} \quad \text{—— (11)}$$

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 9Be^{-3x} \quad \text{—— (111)}$$

(11) + (111)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 6\left(\underbrace{A\cdot e^{2x} + Be^{-3x}}\right)$$

$$= 6y$$

[Using (1)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

$$y_2 + y_1 - 6y = 0$$

$$\begin{vmatrix} y & 1 & 1 \\ y' & 2 & -3 \\ y'' & 4 & 9 \end{vmatrix} = 0$$

$$\begin{vmatrix} y & 1 & 1 \\ y'-2y & 0 & -5 \\ y''-4y & 0 & 5 \end{vmatrix} = 0$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$-1\left[5y' - 10y + 5y'' - 20y\right] = 0$$

$$\Rightarrow 5y'' + 5y' - 30y = 0$$

$$y'' + y' - 6y = 0$$

$$\begin{vmatrix} y & e^{2x} & 1 \cdot e^{-3x} \\ y' & 2e^{2x} & -3 \cdot e^{-3x} \\ y'' & 4e^{2x} & 9 \cdot e^{-3x} \end{vmatrix} = 0$$

$$e^{2x} \cdot e^{-3x} \begin{vmatrix} y & 1 & 1 \\ y' & 2 & -3 \\ y'' & 4 & 9 \end{vmatrix} = 0$$

e^{-x}

(I) divided by (II)

$$\frac{y}{y_1} = \frac{1}{2}(x-a)$$

$$\Rightarrow \frac{2y}{y_1} = x - a$$

$$\Rightarrow a = x - \frac{2y}{y_1}$$

put in (1)

$$y = \left(x - \frac{2y}{y_1}\right)\left(\cancel{x} - \cancel{x} + \frac{2y}{y_1}\right)^2$$

$$\Rightarrow y = \left(\frac{y_1 \cdot x - 2y}{y_1}\right)\left(\frac{4y^2}{y_1^2}\right)$$

<u>Prob</u>: Find the diff. eqn. of the family of curves $y = a(x-a)^2$, where a is an arb. const.

<u>Solⁿ</u>:

$$y = a(x-a)^2 \quad \text{— (I)}$$

$$y_1 = 2a(x-a) \quad \text{— (II)}$$

Ans: $y_2 - 2ay_1 + (a^2+b^2)y = 0$

$\Rightarrow \cancel{y} \cdot y_1^3 = (y_1 x - 2y) \cdot 4y^{\cancel{2}}$

$\Rightarrow y_1^3 = 4xyy_1 - 8y^2$

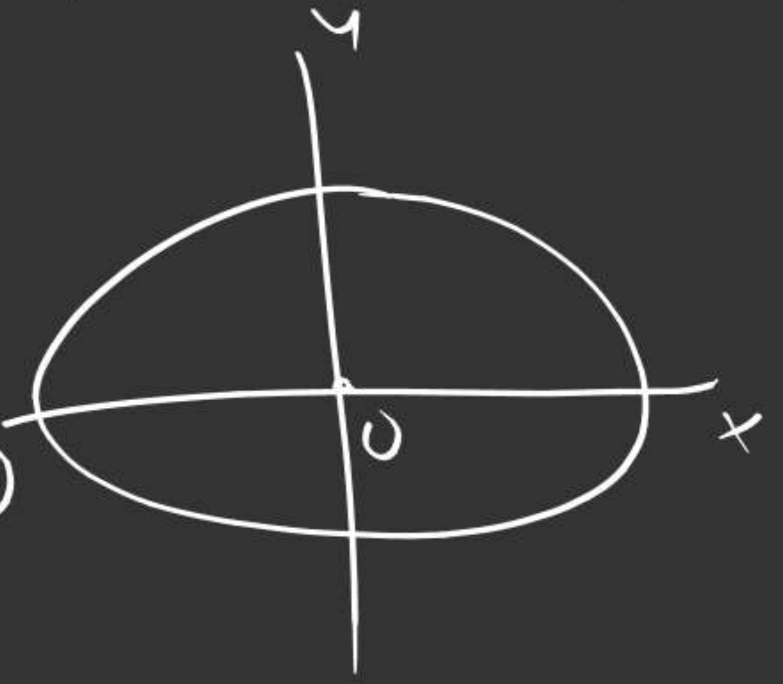
$\Rightarrow y_1^3 - 4xyy_1 + 8y^2 = 0$

Prob: Form the diff. eqn. of $y = e^{ax}(c_1 \sin bx + c_2 \cos bx)$, where c_1 & c_2 are arb. constants.

Prob: Find the diff. eqn. of the family of ellipses whose axes coincide with the axes of co-ordinates and centres at the origin.

Soln:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{--- (1)}$$



Diff. (1), w.r.t. x

$$\frac{2x}{a^2}+\frac{2yy_1}{b^2}=0$$

$$\Rightarrow \frac{x}{a^2}+\frac{yy_1}{b^2}=0$$

$$\Rightarrow \frac{x}{a^2}=-\frac{yy_1}{b^2}$$

$$\Rightarrow -\frac{b^2}{a^2}=\frac{yy_1}{x}$$

Diff. again,

$$0=\frac{x\left[y_1^2+yy_2\right]-yy_1}{x^2}$$

<u>Solⁿ:</u> $x^2+y^2+2ax+2by+c=0$ —— ①

Diff ① w.r.t. x

$\cancel{2}x+\cancel{2}yy_1+\cancel{2}a+\cancel{2}by_1=0$

Diff. it again w.r.t. x

$1+y_1^2+yy_2+by_2=0$

$\Rightarrow \dfrac{1+y_1^2+yy_2}{y_2}=-b$

$x\left[y_1^2+yy_2\right]-yy_1=0$

$\Rightarrow xyy_2+xy_1^2-yy_1=0$

<u>prob:</u> Find the diff. eqn. of the family of circles, given by $x^2+y^2+2ax+2by+c=0$ where a, b, c are arb. constant.

$$3y_1(y_2)^2 - (1+y_1^2)y_3 = 0$$

$$\Rightarrow (1+y_1^2)y_3 = 3y_1(y_2)^2$$

Diff. again w.r.t. x

$$\frac{y_2\left[2y_1y_2 + y_1y_2 + yy_3\right] - (1+y_1^2+yy_2)\,y_3}{(y_2)^2} = 0$$

$$\Rightarrow 3y_1(y_2)^2 + \cancel{yy_2}y_3 - y_3 - y_1^2y_3 - \cancel{yy_2}y_3 = 0$$

Eliminating a, b from (I), (II) & (III).

$$\begin{vmatrix} y & -x^2 & -x \\ y_1 & -2x & -1 \\ y_2 & -2 & 0 \end{vmatrix} = 0$$

$\Rightarrow y_2(x^2 - 2x^2) + 2(-y + x y_1) = 0$

$\Rightarrow -x^2 y_2 + 2x y_1 - 2y = 0$ (Exp. through R_3)

$\Rightarrow x^2 y_2 - 2x y_1 + 2y = 0$

Prob: $y = ax^2 + bx$, a, b are arb. constants.

Sol^n: $y = ax^2 + bx$

$\Rightarrow y - ax^2 - bx = 0$ — (I)

Diff. w.r.t. x,

$y_1 - 2ax - b = 0$ — (II)

Diff. w.r.t. x,

$y_2 - 2a = 0$ — (III)

$x^2+y^2-2ay=0$ — (1)

Diff. w.r.t. x,

$\cancel{2}x+\cancel{2}yy_1-\cancel{2}ay_1=0$

$x+yy_1=ay_1$

$\Rightarrow a=\dfrac{x+yy_1}{y_1}$

Put a in (1)

$x^2+y^2-2\left(\dfrac{x+yy_1}{y_1}\right)\cdot y=0$

Prob: Find the diff. eqn. of the family of circles touching x-axis at the origin.

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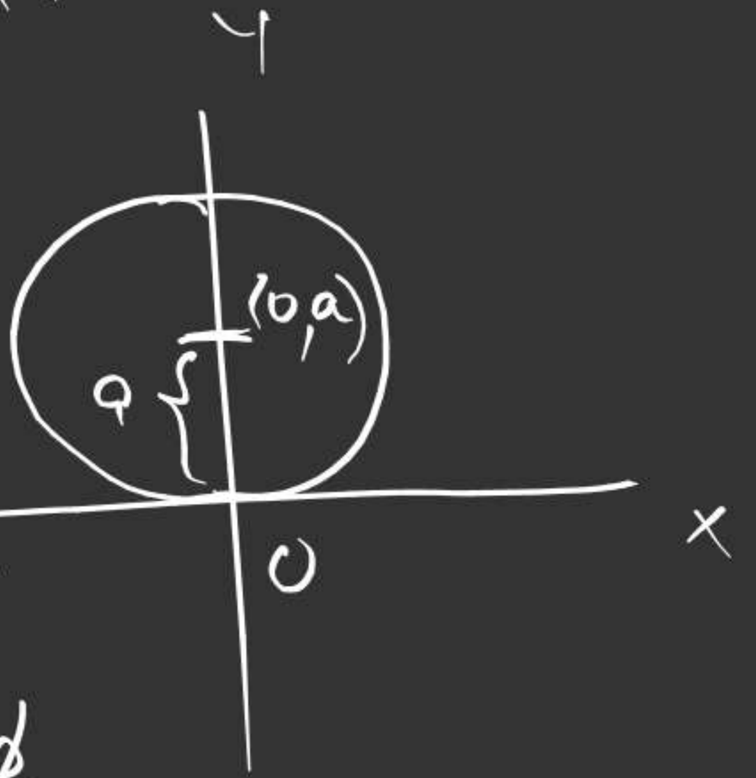

$(x-0)^2+(y-a)^2=a^2$

$x^2+y^2-2ay+\cancel{a^2}=\cancel{a^2}$