KGS IAS

Maths Optional

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Prob: A and B are two $n \times n$ matrices with the same set of 'n' distinct eigen values. Show that there exists two matrices P and Q (one of them is non-singular) s.t. $A = PQ$, $B = QP$.

Sol^n: $\because$ A and B have the same set of 'n' distinct eigen values.

$\therefore$ A and B are similar.

$\therefore \exists$ a non-singular matrix P s.t.

$$P^{-1}AP = B$$

let $Q = P^{-1}A$

$$\therefore B = QP.$$

Also $Q = P^{-1}A$

$$\Rightarrow PQ = P(P^{-1}A) = (PP^{-1})A = A$$

Prob: prove that if A is similar to a Diagonal matrix then A^T is similar to A.

Sol^n: suppose A is similar to a Diagonal matrix D.

$\therefore \exists$ a non-singular matrix P s.t.

$$P^{-1}AP = D$$

$$\Rightarrow A = PDP^{-1}$$

$$\Rightarrow A^T = (PDP^{-1})^T$$

$$= (P^{-1})^T D^T P^T$$

$$= (P^T)^{-1} D \, P^T$$

$[\because D^T = D]$

$\Rightarrow A^T$ is similar to D

and also D is similar to A

$\Rightarrow A^T$ is similar to A.

(transitivity)

multiplicity of λ:

Algebraic and Geometric multiplicity of a characteristic root:

→ If a square matrix A has a characteristic root λ which repeats 't' times then t is called the algebraic multiplicity of λ.

→ If s is the number of LI eigen vectors corresponding to a ch. root λ then 's' is called the geometric

$\lambda^n = 0$

$\Rightarrow \lambda = 0, 0, \ldots$ — n times.

Alg. mult. of '0'

$= n$.

Ex: O_n

ch. root : '0'.

$|O - \lambda I| = 0$

$$= \begin{vmatrix} -\lambda & 0 & \cdots & 0 \\ 0 & -\lambda & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (-\lambda)^n = 0$

$\Rightarrow (-1)^n \cdot \lambda^n = 0$

$\Rightarrow (1-\lambda)^n = 0$

$\Rightarrow \lambda = \underbrace{1, 1, 1, \ldots}_{n \text{ times}}$

Alg. mult. of '1' is

n.

Ex. I_n

Ch. soln $= \underline{1}$

Alg. multiplicity $= ?$

$|I - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & \cdots & 0 \\ 0 & 1-\lambda & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 1-\lambda \end{vmatrix} = 0$$

Results: (1) The geometric multiplicity of a ch. root can not exceed its algebraic mult.

(2) A square matrix is similar to a Diagonal matrix iff the geometric multiplicity of each of its ch. root is equal to its algebraic multiplicity.

LI. Hence find a Diagonal matrix similar to A.

Sol^n: $|A-\lambda I|=0$

$$\lambda^3-8\lambda^2+17\lambda-10=0$$

$$(\lambda-1)\left[\lambda^2-7\lambda+10\right]=0$$

$$(\lambda-1)(\lambda-2)(\lambda-5)=0$$

$$\lambda=1,2,5.$$

Prob: Show that $A=\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ is similar to a Diagonal matrix. Also find the transforming matrix and Diagonal matrix.

or

Show that the characteristic vectors of $A=\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ are

$$\begin{bmatrix} 1 & -2 & 0 \\ -5 & 2 & 2 \\ 3 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \to -\frac{1}{2}R_1$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -8 & 2 \\ 0 & 8 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + 5R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -8 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

Let X be the eigenvector corr. to $\underline{\lambda = -1}$

$$(A - I)X = 0$$

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 4 & 0 \\ -5 & 2 & 2 \\ 3 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_3$

$x_3 = 4 \times \frac{1}{2}K = 2K$

$$X = \begin{bmatrix} K \\ \frac{1}{2}K \\ 2K \end{bmatrix} = K \begin{bmatrix} 1 \\ \frac{1}{2} \\ 2 \end{bmatrix}$$

$$= K \cdot X_1$$

X_1 is the eigen vector

Corr. to $\lambda = 1$.

$X \neq 0$ and is therefore L.I.

$\therefore$ G.M of 1 = A.M of $1 = 1$.

Rank of coeff matrix $= 2$

No. of L.I. sol

$= 3 - 2 = 1$

$x_1 - 2x_2 = 0$

$-8x_2 + 2x_3 = 0$

$2x_2 = x_1 \Rightarrow x_2 = \frac{1}{2}x_1$

$x_2 = \frac{1}{2}k, \quad x_1 = k$

$2x_3 = 8x_2$

$x_3 = 4x_2$

Rank of coeff. Matrix = 2

No. of L.I. soln = 3 − 2
= 1.

$-x_1 + 2x_2 - 2x_3 = 0$

$-12x_2 + 12x_3 = 0$

$12x_2 = 12x_3$

$\Rightarrow x_2 = x_3 = K$

$x_1 = 2x_2 - 2x_3$
$= 2K - 2K$
$= 0$

Let X be the eigen vector corr. to $\lambda = 5$

$$(A - 5I)X = 0$$

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} X = 0$$

$$\begin{bmatrix} -1 & 2 & -2 \\ 0 & -12 & 12 \\ 0 & 0 & 0 \end{bmatrix} X = 0$$

$R_2 \to R_2 - 5R_1$
$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 6 & -3 \\ 0 & 6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + 5R_1$

$R_3 \to R_3 + 2R_1$

$\underline{K \cdot X_2}$

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector (LI) corr. to $\lambda = 5$

AM of 5 = GM of $5 = 1$

Let X be the eigenvector corr. to $\lambda = 2$

$(A - 2I)X = 0$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 - R_2$

Rank of coeff. matrix $= 2$

No. of L.I. sol^n $= 3-2 = 1$

$x_1 + x_2 - x_3 = 0$

$6x_2 - 3x_3 = 0$

$6x_2 = 3x_3$

$\Rightarrow x_2 = \frac{1}{2}x_3$

Let $x_3 = K$

$x_2 = \frac{1}{2}K$

$x_1 = -x_2 + x_3$

$= -\frac{1}{2}K + K$

$= \frac{1}{2}K$

$$X = \begin{bmatrix} \frac{1}{2}K \\ \frac{1}{2}K \\ K \end{bmatrix} = K \underline{\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}} = K \cdot X_3$$

Transforming matrix

$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2}{2} & 1 & 1 \end{bmatrix}$$

All the columns of P are L.I.

$$P^{-1}AP = \text{Diag.}[1, 5, 2]$$

$\therefore \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ is the L.I. eigen vector corr. to $\lambda = 2$.

AM of 2 = GM of 2 = 1

$\therefore$ AM of each of the ch. root = G.M of each the ch. root.

$\therefore$ A is similar to Diagonal matrix.

$\lambda = -1, -1, 3.$

Let X be the eigenvector corr. to $\lambda = -1$

$$\therefore (A + I)X = 0$$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Prob: Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Also find the diagonal form and a diagonalizing matrix P.

Soln.

$$|A - \lambda I| = 0$$

$$(1+\lambda)(-1-\lambda)(3-\lambda) = 0$$

$-8x_1 + 4x_2 + 4x_3 = 0$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$

Let $x_2 = k_1$, $x_3 = k_2$

$\Rightarrow 2x_1 = x_2 + x_3$

$\Rightarrow x_1 = \frac{1}{2}[x_2 + x_3]$

$= \frac{1}{2}[k_1 + k_2]$

$$X = \begin{bmatrix} \frac{1}{2}k_1 + \frac{1}{2}k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$= k_1 X_1 + k_2 X_2$

$$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

Rank of coeff. matrix $= 1$

$\therefore$ No. of L.I. soln

$= 3 - 1 = 2$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_1 \to -\frac{1}{12}R_1$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{8}{3} & \frac{4}{3} \\ 0 & \frac{8}{3} & -\frac{4}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + 8R_1$
$R_3 \to R_3 + 16R_1$

$\therefore X_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$

are L.I. eigen vectors. Corr. to $\lambda = -1$.

GM of $(\lambda = -1) = 2 =$ AM of $\lambda = -1$

Let X be the eigen vector Corr. to $\lambda = 3$.

$$(A - 3I)X = 0$$

$\Rightarrow -8x_2 + 4x_3 = 0$

$\Rightarrow -2x_2 + x_3 = 0$

$\Rightarrow x_3 = 2x_2$

let $x_2 = K$

$\underline{x_3 = 2K}$

$3x_1 - x_2 - x_3 = 0$

$\Rightarrow 3x_1 - K - 2k = 0$

$\Rightarrow x_1 = K.$

$$\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{8}{3} & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + R_2$

Rank of coeff. matrix $= 2$

No. of LI Sol^n. $= 3 - 2 = 1$

$x_1 - \frac{1}{3}x_2 - \frac{1}{3}x_3 = 0$

$-\frac{8}{3}x_2 + \frac{4}{3}x_3 = 0$

$\therefore$ AM of each of ch. roots is equal to their GM.

$\therefore$ A is Diagonalizable

$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$P^{-1}AP = \mathrm{Diag}[-1, -1, 3]$$

$$\therefore X = \begin{bmatrix} K \\ K \\ 2K \end{bmatrix} = K \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = K \cdot X_3$$

$X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigen vector (L I) corr. to $\lambda = 3$

AM of $(\lambda = 3)$ = G.M. of $(\lambda = 3)$
$= 1$

Prob: Show that the following matrix is not similar to a Diagonal matrix:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hint: $\lambda = 2, 2, 1$