KGS IAS

Maths Optional

By Dhrub Singh Sir

$v_1, v_2 — v_n$ — L.I.

$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$

$a_i \in F$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$

$a(1,0,0) + b(0,1,0) + c(0,0,1) = 0$

$\Rightarrow a = b = c = 0$

$v_1, v_2, \ldots v_n$

L.D.

$\exists\ a_1, a_2, — a_n \in F$
not all zeroes.
s.t

$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$

$v_1 = (1, 2, 3) \quad v_2 = (2, 4, 6)$

$-2 v_1 + v_2 = (0, 0, 0)$

Result: A set consisting of single non-zero vector is L.I.

proof: Let $S=\{v\}\subseteq V(F)$

$v\neq 0$

$av=0,\quad a\in F$

$\because v\neq 0$

$\therefore av=0$

$\Rightarrow a=0$

$\therefore S$ is L.I.

Note ① Any set S of vectors in V is said to be LI(LD) if every finite subset of S is LI(LD).

(ii) The set $\{0\}$ is L.D.

If $\alpha \neq 0$, $\therefore \alpha^{-1}$ exists

(1) $\Rightarrow$

$$\alpha^{-1}(\alpha u + \beta v) = \alpha^{-1} \cdot 0 = 0$$

$$\Rightarrow (\alpha^{-1}\alpha) u + (\alpha^{-1}\beta) v = 0$$

$$\Rightarrow 1 \cdot u = -(\alpha^{-1}\beta) v$$

$$\Rightarrow u = -(\alpha^{-1}\beta) v$$

<u>Result</u>: If two vectors are L.D. then one of them is a scalar multiple of the other.

<u>Proof</u>: Let $V(F)$ be a vector space.

Let u, v are L.D. vectors.

$\therefore \exists$ scalars α, β (not all zero) s.t.

$$\alpha u + \beta v = 0 \quad \text{—(1)}$$

Result: If the sets $\{v_1, v_2, - v_n\}$ of the vectors of $V(F)$ is L.I. then none of the vectors can be a zero vector.

Proof: $\because$ S is L.I.

$\therefore \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0, \alpha_i \in F$

Let, if possible, $\exists$ some

If $\beta \neq 0$, $\therefore \beta^{-1}$ exists

$(1) \Rightarrow$

$\beta^{-1}(\alpha u + \beta v) = \beta^{-1} \cdot 0 = 0$

$\Rightarrow (\beta^{-1}\alpha) u + \underline{(\beta^{-1}\beta)} v = 0$

$\Rightarrow 1 \cdot v = -(\beta^{-1}\alpha) u$

$\Rightarrow v = -(\beta^{-1}\alpha) u$

Result: A set of vectors which contains the zero vector is L.D.

Proof: Let $V(F)$ be a vector space.

Let $S = \{v_1, v_2, \dots v_n\} \subseteq V$

and $v_k = 0$, for some k.

Consider the linear combination

$0 \cdot v_1 + 0 \cdot v_2 + \dots + \alpha_k \cdot v_k + 0 \cdot v_{k+1} + \dots + 0 \cdot v_n = 0$

$\underline{\alpha_k \neq 0}$

$v_k = 0 \quad (1 \leq k \leq n)$

Then we can have.

$0 \cdot v_1 + 0 \cdot v_2 + \dots - + \alpha_k \cdot v_k + 0 \cdot v_{k+1} + \dots + 0 \cdot v_n = 0$

$\Rightarrow \{v_1, v_2, \text{—} v_n\}$ is L.D. $\quad \alpha_k \neq 0 \in F$.

Which is a contradiction.

$\therefore$ our supposition is wrong.

$\therefore v_k \neq 0 \quad \forall k$
$1 \leq k \leq n$.

Consider $S' = \{v_1, v_2, \ldots v_r\}$.

$1 \le r \le n$

$\therefore S' \subseteq S$

Consider
$a_1 v_1 + a_2 v_2 + \cdots + a_r v_r = 0$

So, we can have $\quad a_i \in F$

$a_1 v_1 + a_2 v_2 + \cdots + a_r v_r + 0 \cdot v_{r+1}$
$+ \cdots + 0 \cdot v_n = 0$

$\Rightarrow a_1 = a_2 = \cdots = a_r = 0$
(As S is L.I)
$\therefore S'$ is L.I.

Here not all scalars are zeros.

$\therefore \{v_1, v_2, \text{---} v_n\}$ is L.D.

Th: A subset of a L.I. set is L.I.

Proof: Let $V(F)$ be a vector space.
Let $S = \{v_1, v_2, \text{---} v_n\}$ is L.I.

To prove: T is L.D.

As S is L.D.

$\therefore \exists$ some scalars $a_1, a_2, \ldots a_r$ (not all zeros) s.t

$$a_1 v_1 + a_2 v_2 + \cdots + a_r v_r = 0$$

$$\Rightarrow a_1 v_1 + a_2 v_2 + \cdots + a_r v_r + 0 \cdot v_{r+1} + \cdots + 0 \cdot v_n = 0$$

$\Rightarrow$ T is L.D.
(some a_i's are non-zero)

Th: A superset of a L.D. set is L.D.

Proof: Let V(F) be a vector space.

Let $S = \{v_1, v_2, \ldots v_r\} \subseteq V$ is a L.D. set.

Let $T = \{v_1, v_2, \ldots v_r, v_{r+1}, \ldots v_n\}$

$T \supseteq S$.

$a+b+2c=1$

$a+2b-c=-2$

$a+3b+c=5$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

Prob: Write the vector $v=(1,-2,5)$ as a linear combination of the elements of the set

$$\{(1,1,1),(1,2,3),(2,-1,1)\} \subseteq \mathbb{R}^3.$$

Solⁿ $(1,-2,5) = a(1,1,1) + b(1,2,3) + c(2,-1,1)$

$a, b, c \in \mathbb{R}$

$a+b+2c=1$

$\Rightarrow a+3+2\times 2=1$

$\Rightarrow a+7=1$

$a=-6$

$\therefore v=-6(1,1,1)+3(1,2,3)$
$+2(2,-1,1)$

$$\sim\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$r(A) = r(A:B) = 3$
$=$ No. of var.

unique soln.

$5c=10 \Rightarrow c=2$

$b-3c=-3$

$\Rightarrow b-6=-3$

$\Rightarrow b=3$

$$\begin{bmatrix} 4 & 2 \\ 3 & -6 \\ -2 & 7 \end{bmatrix}_{3\times 2}$$

$$\begin{vmatrix} 4 & 2 \\ 3 & -6 \end{vmatrix} = -24 - 6$$
$$= -30 \neq 0$$

$\therefore$ Rank of coeff. matrix $= 2 =$ No. of var.

only soln.

$a = b = 0$

$\therefore u$ & v are L.I.

Prob: Determine whether u & v are L.D.

(i) $u = (4, 3, -2)$, $v = (2, -6, 7)$

(ii) $u = (-4, 6, -2)$, $v = (2, -3, 1)$

Soln: (i)

$a(4, 3, -2) + b(2, -6, 7) = (0, 0, 0)$

$\Rightarrow 4a + 2b = 0$

$3a - 6b = 0$

$-2a + 7b = 0$

(II) $u = -2v$.

$\therefore$ u & v are L.D.

If u & v (or) are L.D. then
$u = av$

$(4, 3, -2) = a(2, -6, 7)$

$\Rightarrow 4 = 2a \Rightarrow a = 2$

$3 = -6a \Rightarrow a = -\frac{1}{2}$

$-2 = 7a \Rightarrow a = -\frac{2}{7}$

Diff. $\underline{a}$.

$\therefore$ u and v are not L.D.

Proof: $S = \{v_1, v_2, \ldots v_n\} \subseteq V$

and S is L.D.

Also, S contains non-zero vectors.

$\because$ S is L.D.

$\therefore$ $\exists$ some scalars

$a_1, a_2, \ldots a_n$ (not all zeros) s.t.

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0 \quad \text{—} (1)$$

Th: Let $V(F)$ be a vector space and $S = \{v_1, v_2, \ldots v_n\} \subseteq V$ (contains non-zero vectors) and if S is LD then one of the vectors of S, say v_i $(1 < i \leq n)$ is a linear combination of its preceding vectors.

It contradicts the hypothesis that S has some non-zero elements.

$\therefore i \neq 1$.

so $1 < i \leq n$

Now (1) $\Rightarrow$

$$a_1 v_1 + a_2 v_2 + \cdots + a_i v_i = 0$$

$$\Rightarrow a_i v_i = -a_1 v_1 - a_2 v_2 - \cdots - a_{i-1} v_{i-1}$$

The maximal value of K, say i s.t.

$$a_i \neq 0$$

i.e. $a_{i+1} = a_{i+2} = \cdots = a_n = 0$

If $a_i = a_1$,

(1) $\Rightarrow$ $a_1 v_1 = 0$

$$\Rightarrow v_1 = 0 \quad (a_1 \neq 0)$$

$$\Rightarrow a_i^{-1}(a_i v_i) = -(a_i^{-1}a_1 v_1 + a_i^{-1}a_2 v_2 + \cdots + a_i^{-1} \cdot a_{i-1} v_{i-1})$$

$\left[\because a_i \neq 0 \therefore a_i^{-1} \text{ exists}\right]$

$$\Rightarrow v_i = -(a_i^{-1}a_1 v_1 + \cdots + a_i^{-1}a_{i-1} v_{i-1})$$

$\left[\because a_i^{-1} \cdot a_i = 1\right]$

Th: Let $V(F)$ be the vector space and $S = \{v_1, v_2, \dots v_n\} \subseteq V$ (contains non-zero vectors also). If one of the vectors of 'S' say v_i $(1 < i \leq n)$ is a linear combination of its preceding vectors then S is L.D.

proof: $S = \{v_1, v_2, \dots v_n\} \subseteq V$

Suppose v_i $(1 < i \leq n)$ is a linear comb. of its preceding vectors.

$$v_i = a_1 v_1 + a_2 v_2 + \dots + a_{i-1} v_{i-1}$$

$$a_i \in F.$$

$$\Rightarrow a_1 v_1 + a_2 v_2 + \dots + a_{i-1} v_{i-1} + (-1) v_i + 0 \cdot v_{i+1} + \dots + 0 \cdot v_n = 0$$

Sol^n: Observe that

$$(1,2,4) = 1\cdot(1,0,0) + 2(0,1,0) + 4\cdot(0,0,1)$$

And so, S is L.D.

$\therefore S = \{u_1, u_2, \text{—} u_n\}$ is L.D.

as $a_i = -1 \neq 0$.

Prob: Show that

$S = \{(1,2,4), (1,0,0), (0,1,0), (0,0,1)\}$ is a L.D subset of $\mathbb{R}^3$.

$a+2b+2c=3$ — (I)

$-2a+3b-c=9$ — (II)

$2c=-4 \Rightarrow c=-2$

$3a-b+c=-2$ — (III)

put $c=-2$ in (I) & (II),

$a+2b=7$ — (IV)

$-2a+3b=7$ — (V)

$2\times$(IV) $+$ (V)

prob: Determine whether or not $v=(3,9,-4,-2)$ in $\mathbb{R}^4$ is a linear combination of $v_1=(1,-2,0,3)$, $v_2=(2,3,0,-1)$ & $v_3=(2,-1,2,1)$.

Solⁿ:

$(3,9,-4,-2)=a(1,-2,0,3)$

$+b(2,3,0,-1)+c(2,-1,2,1)$

$$\therefore \boxed{v = v_1 + 3v_2 - 2v_3}$$

$$\cancel{2}a + 4b = 14$$
$$-\cancel{2}a + 3b = 7$$

$$7b = 21$$

$$b = 3$$

Putting $b = 3$ in (iv),

$$a = 1$$

$a = 1, b = 3, c = -2$ also satisfy (iii),

$a(-1,2,1)+b(3,0,-1)$
$+c(-5,4,3)$
$=(0,0,0)$

$\Rightarrow -a+3b-5c=0$

$2a+0\cdot b+4c=0$

$a-b+3c=0$

$$\begin{bmatrix} -1 & 3 & -5 \\ 2 & 0 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

Prob: Prove that the set

$S=\{\overset{v_1}{(-1,2,1)}, \overset{v_2}{(3,0,-1)}, \overset{v_3}{(-5,4,3)}\}$

$\subseteq \mathbb{R}^3$ is L.D.

Sol^n: observe that

$V_3 = 2v_1 - v_2$

$\therefore \{v_1, v_2, v_3\}$ is L.D

$\therefore$ The system has non-zero sol^n also.

$\therefore$ S is L.D.

$$\sim \begin{bmatrix} -1 & 3 & -5 \\ 0 & 6 & -6 \\ 0 & 2 & -2 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} -1 & 3 & -5 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{3} R_2$

$\therefore$ Rank of coeff matrix $= 2$
$<$ No. of var.

Prob: Examine the vectors $(1, 1, 2, 4)$, $(2, -1, -5, 2)$, $(1, -1, -4, 0)$, $(2, 1, 1, 6)$ are L.I. in $\mathbb{R}^4$.

HW

Ans: L.D.