KGS IAS

Maths Optional

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$$C.F. = e^{x}\left[c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right]$$

HW

prob:

$$(D^2-2D+3)y = x^2+\cos x$$

Soln:

A.E.

$$m^2-2m+3=0$$

$$m = \frac{2\pm\sqrt{4-12}}{2}$$

$$= \frac{2\pm\sqrt{-8}}{2}$$

$$= \frac{2\pm 2\sqrt{2}i}{2}$$

$$= 1\pm\sqrt{2}i$$

$$\underline{PI} = \frac{1}{D^2-2D+3}\cos x + \frac{1}{D^2-2D+3}x^2$$

$$= \frac{1}{-1^2-2D+3}\cos x + \frac{1}{3}\left[1+\frac{D^2-2D}{3}\right]^{-1}x^2$$

$$= \frac{1}{-2D+2}\cos x + \frac{1}{3}\left[1-\frac{D^2-2D}{3} + \frac{(D^2-2D)^2}{9}\right]x^2$$

$$= -\frac{1}{2}\frac{D+1}{(D-1)(D+1)}\cos x + \frac{1}{3}\left[1+\frac{2D}{3}-\frac{D^2}{3}+\frac{4D^2}{9}\right]x^2$$

Case IV $\phi = e^{ax} \cdot V$, where V is a fn. of 'x'.

$$f(D)y = \phi.$$

$$\boxed{P.I. = \frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} \cdot V}$$

$$= -\frac{1}{2} \frac{(D+1)}{D^2-1} \cos x + \frac{1}{3}\left[1 + \frac{2D}{3} + \frac{D^2}{9}\right] x^2$$

$$= -\frac{1}{2} \frac{(D+1)\cos x}{-2} + \frac{1}{3}\left[x^2 + \frac{4x}{3} + \frac{2}{9}\right]$$

$$= \frac{1}{4}\left[\cos x - \sin x\right] + \frac{1}{3}\left[x^2 + \frac{4x}{3} + \frac{2}{9}\right]$$

Gen. soln. is

$$y = CF + PI.$$

$$= 1 \pm \sqrt{3}\, i$$

$$C.F. = e^{x}\left[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right]$$

$$P.I. = \frac{1}{D^2 - 2D + 4}\, e^{x} \cdot \cos x$$

$$= e^{x}\, \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^{x}\, \frac{1}{D^2 + 2\cancel{D} + 1 - 2\cancel{D} - 2 + 4} \cos x$$

prob: Solve

$$(D^2 - 2D + 4)y = e^{x} \cos x$$

A.E. $\quad m^2 - 2m + 4 = 0$

$$m = \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}\, i}{2}$$

Prob: solve

$$(D^2-1)y = e^x(1+x^2)$$

Solⁿ: A.E. is

$$m^2-1=0$$

$$\Rightarrow m = \pm 1$$

$$CF = c_1 e^x + c_2 e^{-x}$$

$$PI = \frac{1}{D^2-1} e^x (1+x^2)$$

$$= e^x \frac{1}{D^2+3} \cos x$$

$$= e^x \frac{1}{-1^2+3} \cos x$$

$$= \frac{e^x \cos x}{2}$$

$\therefore$ Gen. solⁿ. is

$$y = CF + PI.$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left[1 + x^2 - x + \frac{1}{2}\right]$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left[x^2 - x + \frac{3}{2}\right]$$

$$= \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right]$$

$$= \frac{e^x}{2} \cdot x\left[\frac{2x^2 - 3x + 9}{6}\right]$$

$$= \frac{e^x}{12} \cdot x(2x^2 - 3x + 9)$$

G.S. is $y = CF + PI$

$$= e^x \frac{1}{(D+1)^2 - 1}(1 + x^2)$$

$$= e^x \frac{1}{D^2 + 2D + \cancel{1} - \cancel{1}}(1 + x^2)$$

$$= e^x \cdot \frac{1}{2D}\left[1 + \frac{D}{2}\right]^{-1}(1 + x^2)$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left[1 - \frac{D}{2} + \frac{D^2}{4}\right](1 + x^2)$$

$$\underline{PI} = \frac{1}{D^2+2} x^2 e^{3x} + \frac{1}{D^2+2} e^{x} \cos 2x$$

$$\frac{1}{D^2+2} x^2 e^{3x} = e^{3x} \frac{1}{(D+3)^2+2} x^2$$

$$= e^{3x} \cdot \frac{1}{D^2+6D+9+2} x^2$$

$$= e^{3x} \frac{1}{D^2+6D+11} x^2$$

$$= \frac{e^{3x}}{11} \left[1 + \frac{D^2+6D}{11}\right]^{-1} x^2$$

<u>prob:</u> Solve

$$(D^2+2)y = x^2 e^{3x} + e^{x} \cos 2x$$

<u>Sol^n:</u> A.E.

$$m^2+2=0$$

$$\Rightarrow m^2 = -2$$

$$\Rightarrow m = \pm\sqrt{2}\, i$$

$$C.F. = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x.$$

$$\frac{1}{D^2+2} e^{x} \cos 2x$$

$$= e^{x} \frac{1}{(D+1)^2+2} \cos 2x$$

$$= e^{x} \frac{1}{D^2+2D+3} \cos 2x$$

$$= e^{x} \frac{1}{-4+2D+3} \cos 2x$$

$$= e^{x} \frac{1}{2D-1} \cos 2x$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{D^2+6D}{11} + \frac{(D^2+6D)^2}{121}\right]x^2$$

$$= \frac{e^{3x}}{11}\left\{1 - \frac{6D}{11} - \frac{D^2}{11} + \frac{36D^2}{121}\right\}x^2$$

$$= \frac{e^{3x}}{11}\left\{1 - \frac{6D}{11} + \frac{25D^2}{121}\right\}x^2$$

$$= \frac{e^{3x}}{11}\left\{x^2 - \frac{12}{11}x + \frac{50}{121}\right\}$$

$\therefore$ gen. soln.

$$y = CF + P.I.$$

$$= e^{x}\,\frac{2D+1}{4D^2-1}\cos 2x$$

$$= \frac{e^{x}}{-17}\,(2D+1)\cos 2x$$

$$= \frac{e^{x}}{-17}\left(\cos 2x - 4\sin 2x\right)$$

$$PI = \frac{e^{3x}}{11}\left[x^2 - \frac{12}{11}x + \frac{S}{12},\right)$$

$$-\frac{e^{x}}{17}\left[\cos 2x - 4\sin 2x\right]$$

Prob: Solve

$$(D^2+4)y = x\sin x$$

Sol^n:

A.E

$$m^2+4=0$$

$$\Rightarrow m^2=-4$$

$$\Rightarrow m=\pm 2i$$

$$CF = c_1\cos 2x + c_2\sin 2x.$$

$$PI = \text{Imaginary part of } \frac{1}{D^2+4} e^{ix}\cdot x$$

Now $\frac{1}{D^2+4} e^{ix}\cdot x$

$$= e^{ix}\cdot\frac{1}{(D+i)^2+4}x$$

$$= e^{ix}\cdot\frac{1}{D^2+i^2+2iD+4}x$$

$$= \frac{1}{3}\left[\cos x + i\sin x\right]\left[x - \frac{2}{3}i\right]$$

$$PI = \text{Im. part of } \frac{1}{D^2+4} e^{ix} \cdot x$$

$$= \frac{1}{3}\left[x\sin x - \frac{2}{3}\cos x\right]$$

$$= \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

$\therefore$ Gen. soln. is

$$\underline{y = CF + PI}$$

$$= e^{ix} \cdot \frac{1}{D^2+2iD+3} x$$

$$= \frac{e^{ix}}{3}\left[1 + \frac{D^2+2iD}{3}\right]^{-1} x$$

$$= \frac{e^{ix}}{3}\left[1 - \frac{2iD}{3}\right] x$$

$$= \frac{e^{ix}}{3}\left[x - \frac{2}{3}i\right]$$

Prob: Solve

$$\frac{d^2y}{dx^2}+4y=x\sin x$$

Sol^n:

A·E

$m^2+4=0$

$m^2=-4$

$\Rightarrow m=\pm 2i$

$C.f = c_1\cos 2x + c_2\sin 2x$

Case V

$\varphi = x\cdot V,$

V is a fn. of x.

$f(D)y=\varphi$

$$\boxed{P.I. = \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}V}$$

$$= \frac{x\sin x}{3} - \frac{2}{9} D\sin x$$

$$= \frac{x\sin x}{3} - \frac{2\cos x}{9}$$

$\therefore$ Gen. sol^n. is

$$y = CF + PI$$

$$P.I. = \frac{1}{D^2+4} x\sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \cdot \frac{1}{D^2+4}\sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{\sin x}{3}$$

$$= \frac{x\sin x}{3} - \frac{2}{3}\frac{D}{D^2+4}\sin x$$

$$PI = \frac{1}{D^2+4}\, x \sin 2x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{D^2+4} \sin 2x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \left(\frac{-x\cos 2x}{4}\right)$$

$$= \frac{-x^2 \cos 2x}{4} + \frac{\cancel{2}D}{\underset{2}{\cancel{4}}(D^2+4)} (x\cos 2x)$$

prob: Solve

$$(D^2+4)y = x \cdot \sin 2x$$

Soln:

A.E.

$$m^2+4=0$$

$$\Rightarrow m = \pm 2i$$

$$C.F. = c_1 \cos 2x + c_2 \sin 2x$$

$$2PI = \frac{-x^2\cos 2x}{4} + \frac{x}{8}\sin 2x$$

$$PI = -\frac{x^2\cos 2x}{8} + \frac{x\sin 2x}{16}$$

$$PI = -\frac{x^2\cos 2x}{4} + \frac{1}{2}\cdot\frac{1}{D^2+4}\left[\cos 2x - 2x\sin 2x\right]$$

$$= -\frac{x^2\cos 2x}{4} + \frac{1}{2}\cdot\frac{1}{D^2+4}\cos 2x$$

$$\underbrace{-\frac{1}{D^2+4}x\sin 2x}_{PI}$$

$$\Rightarrow 2\,P.I. = \frac{-x^2\cos 2x}{4} + \frac{1}{2}\cdot\frac{x}{2}\cdot\frac{\sin 2x}{2}$$

$$PI = \frac{1}{D^2+1} x^2 \sin 2x$$

$$= \text{Im. part of}$$

$$\frac{1}{D^2+1} e^{i2x} \cdot x^2$$

Now $\frac{1}{D^2+1} e^{2ix} \cdot x^2$

Note: If $\phi = x^m \sin ax$

or $x^m \cos ax$

$m \geq 2$

Ex: Find P.I. of

$$(D^2+1)y = x^2 \sin 2x$$

$$= e^{2ix} \cdot \frac{1}{(D+2i)^2+1} x^2$$

$$= e^{2ix} \cdot \frac{1}{D^2+4iD-4+1} x^2$$

$$= e^{2ix} \cdot \frac{1}{D^2+4iD-3} x^2$$

$$= -\frac{e^{2ix}}{3}\left[1-\frac{D^2+4iD}{3}\right]^{-1} x^2$$

$$= -\frac{e^{2ix}}{3}\left[1+\frac{D^2+4iD}{3}+\frac{(D^2+4iD)^2}{9}\right] x^2$$

$$= -\frac{e^{2ix}}{3}\left[1+\frac{4iD}{3}+\frac{D^2}{3}-\frac{16D^2}{9}\right] x^2$$

$PI = \text{Im. part}$

$$= -\frac{1}{3}\left[\frac{8}{3}x\cos 2x + \left(x^2 - \frac{26}{9}\right)\sin 2x\right]$$

$$= -\frac{8}{9}x\cos 2x - \frac{x^2}{3}\sin 2x + \frac{26}{27}\sin 2x$$

$$= \frac{-e^{2ix}}{3}\left[1 + \frac{4iD}{3} - \frac{13D^2}{9}\right]x^2$$

$$= -\frac{e^{2ix}}{3}\left[x^2 + i\frac{8}{3}x - \frac{26}{9}\right]$$

$$= -\frac{1}{3}\left[\cos 2x + i\sin 2x\right]$$

$$\left[x^2 - \frac{26}{9} + i\cdot\frac{8}{3}x\right]$$