Maths Optional

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Cauchy-Euler Eqn:

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X \quad \text{---} (1)$$

$a_1, a_2, \ldots a_n$ — constants.

X — fn. of 'x'.

Put $x = e^z \iff z = \log x$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{1}{x} \frac{dy}{dz}$$

$$x \frac{dy}{dx} = \frac{dy}{dz}$$

$$x D y = D_1 y, \quad D_1 = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

$$=\left(D_1^2-D_1\right)y$$

$$x^2D^2y=D_1(D_1-1)y$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x^3D^3y=D_1(D_1-1)(D_1-2)y$$

$$\vdots$$

$$x^nD^ny=D_1(D_1-1)(D_1-2)\cdots(D_1-(n-1))y.$$

$$\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\times\frac{dy}{dz}\right)$$

$$=-\frac{1}{x^2}\cdot\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$=-\frac{1}{x^2}\frac{dy}{dx}+\frac{1}{x^2}\frac{d^2y}{dz^2}$$

$① \Rightarrow$

$$\left[D_1(D_1-1)-D_1+2\right]y = e^{z}\cdot z$$

$D_1 = \frac{d}{dz}$

$$\Rightarrow \left[D_1^2 - D_1 - D_1 + 2\right]y = z\cdot e^{z}$$

$$\Rightarrow \left[D_1^2 - 2D_1 + 2\right]y = ze^{z}$$

A.E. $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{-4}}{2}$$

$$= 1 \pm i$$

Putting all these in ①,

① will become L.D.E with constant coeffs involving 'y' and 'z'.

<u>Prob:</u> Solve $(x^2D^2 - xD + 2)y = x\log x$ — ①

Put $x = e^{z}$

$\Leftrightarrow z = \log x$

$$= e^{z} \cdot \frac{1}{D_1^2+1} z$$

$$= e^{z} \left(1+D_1^2\right)^{-1} z$$

$$= e^{z} (1) z$$

$$= e^{z} \cdot z$$

$$= x \cdot \log x$$

$$\therefore y = CF + PI$$

$$CF = e^{z}\left[c_1 \cos z + c_2 \sin z\right]$$

$$= x\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right]$$

$$PI = \frac{1}{D_1^2 - 2D_1 + 2} z e^{z}$$

$$= e^{z} \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} z$$

$$= e^{z} \cdot \frac{1}{D_1^2 + \cancel{2D_1} + 1 - \cancel{2D_1} \cancel{-2} \cancel{+2}} z$$

A.E.

$$m^2-5m+6=0$$

$$\Rightarrow m^2-2m-3m+6=0$$

$$\Rightarrow m(m-2)-3(m-2)=0$$

$$\Rightarrow (m-2)(m-3)=0$$

$$m=2,3$$

$$C.F.=c_1e^{2z}+c_2e^{3z}$$

$$=c_1x^2+c_2x^3$$

$$PI=\frac{1}{D_1^2-5D_1+6}e^{z}=\frac{1}{1^2-5+6}e^{z}$$

$$=\frac{e^z}{2}$$

Prob: solve

$$x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=x \quad -(1)$$

Put $x=e^z \Leftrightarrow z=\log x$

$$(1)\Rightarrow$$

$$\left[D_1(D_1-1)-4D_1+6\right]y=e^z$$

$$\Rightarrow \left[D_1^2-D_1-4D_1+6\right]y=e^z$$

$$\Rightarrow \left[D_1^2-5D_1+6\right]y=e^z.$$

$$\left(D_1(D_1-1)-3D_1+4\right)y=(e^z-1)^2$$

$$\Rightarrow \left[D_1^2-D_1-3D_1+4\right]y=e^{2z}-2e^z+1$$

$$\Rightarrow \left[D_1^2-4D_1+4\right]y=e^{2z}-2e^z+1$$

A.E.

$$m^2-4m+4=0$$
$$\Rightarrow (m-2)^2=0$$
$$\Rightarrow m=2,2$$

$$CF=(c_1+c_2z)e^{2z}$$

$$\therefore P.I=\frac{x}{2}$$

General sol^n. is

$$y=CF+P.I.$$

Prob: $(x+1)^2\frac{d^2y}{dx^2}-3(x+1)\frac{dy}{dx}+4y=x^2$ —— (1)

Put $x+1=e^z$

$$\Leftrightarrow z=\ln(x+1)$$

$$= \frac{z^2}{\lfloor 2} e^{2z} - 2e^{z} + \frac{1}{4}$$

$$= \frac{[\log(x+1)]^2}{2} (x+1)^2 - 2(x+1) + \frac{1}{4}$$

$\therefore$ gen soln. is

$$y = CF + PI$$

$$= [c_1 + c_2 \log(x+1)](x+1)^2$$

$$= c_1 (x+1)^2 + c_2 (x+1)^2 . \log(x+1)$$

$$PI = \frac{1}{(D_1 - 2)^2} (e^{2z} - 2e^{z} + 1)$$

$$= \frac{1}{(D_1 - 2)^2} e^{2z} - 2 \cdot \frac{1}{(D_1 - 2)^2} e^{z} + \frac{1}{(D_1 - 2)^2} e^{0z}$$

$$\boxed{\frac{1}{D_1+\alpha}X = x^{-\alpha}\int x^{\alpha-1}\cdot X\,dx}$$

$$\frac{1}{D_1-\alpha}X = U \text{ (say)}$$

$$\Rightarrow (D_1-\alpha)U = X$$

$$\Rightarrow \frac{dU}{dz} - \alpha U = X$$

$$U\cdot e^{-\alpha z} = \int e^{-\alpha z}\cdot X\,dz.$$

Alternative method to find PI

without changing the R.H.S in the fn. of 'z'.

$$\underline{f(D_1)}\,y = \boxed{X} \qquad D_1 \equiv \frac{d}{dz}$$

$$\boxed{\frac{1}{D_1-\alpha}X = x^{\alpha}\int x^{-\alpha-1}\cdot X\,dx} \qquad \boxed{\begin{array}{c} z = \log x \\ \Leftrightarrow x = e^{z} \end{array}}$$

Prob: solve

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x + \sin x \quad (1)$$

$\rightarrow$ Put $x = e^z$

$\Leftrightarrow z = \log x$

(1) $\Rightarrow$

$$[D_1(D_1 - 1) + 4D_1 + 2]y = x + \sin x$$

$$U = e^{\alpha z}\int e^{-\alpha z} \cdot X \, dz$$

$$= x^{\alpha}\int x^{-\alpha} \cdot X \cdot \frac{1}{x} dx$$

$$U = x^{\alpha}\int x^{-\alpha-1} \cdot X \, dx$$

$$\boxed{\frac{1}{D_1 - \alpha} X = x^{\alpha}\int x^{-\alpha-1} \cdot X \, dx}$$

$x = e^z$

$x^{\alpha} = e^{\alpha z}$

dz

$= d\log x$

$= \frac{1}{x} \cdot dx$

$$\underline{PI} = \frac{1}{D_1^2 + 3D_1 + 2}(x + \sin x)$$

$$= \frac{1}{(D_1+1)(D_1+2)} x + \frac{1}{(D_1+1)(D_1+2)} \sin x$$

$$= \frac{1}{D_1+2}\left[\frac{1}{D_1+1} x\right] + \frac{1}{D_1+2}\left[\frac{1}{D_1+1} \sin x\right]$$

$$= \frac{1}{D_1+2}\left[x^{-1}\int x^{1-1} \cdot x \, dx\right]$$

$$+ \frac{1}{D_1+2}\left[x^{-1}\int x^{1-1} \cdot \sin x \, dx\right]$$

$$\left[D_1^2 - D_1 + 4D_1 + 2\right] y = x + \sin x$$

$$\Rightarrow \left[D_1^2 + 3D_1 + 2\right] y = x + \sin x.$$

<u>A.E.</u>

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow (m+1)(m+2) = 0$$

$$\Rightarrow m = -1, -2$$

$$C.F = c_1 e^{-z} + c_2 e^{-2z}$$

$$= \frac{c_1}{x} + \frac{c_2}{x^2}.$$

Prob: Solve

$$(x^2D^2+3xD+1)y=\frac{1}{(1-x)^2} \quad (1)$$

Soln: Put $x=e^z$

$\Leftrightarrow z=\log x$

(1) $\Rightarrow$

$$[D_1(D_1-1)+3D_1+1]y$$

$$\Rightarrow [D_1^2+2D_1+1]y=\frac{1}{(1-x)^2}$$

$$=\frac{1}{(1-x)^2}$$

$$=\frac{1}{D_1+2}\left(\frac{x}{2}\right)+\frac{1}{D_1+2}\left[-\frac{\cos x}{x}\right]$$

$$=\frac{1}{2}\cdot x^{-2}\int x^{2-1}\cdot x\,dx+x^{-2}\int \cancel{x^{2-1}}\left(-\frac{\cos x}{\cancel{x}}\right)dx$$

$$=\frac{1}{2}\cdot\frac{1}{\cancel{x^2}}\cdot\frac{x^{\cancel{3}}}{3}+\frac{1}{x^2}(-\sin x)$$

$$=\frac{x}{6}-\frac{\sin x}{x^2}$$

$\therefore$ Gen. soln. is

$y=CF+PI$.

$$P.I = \frac{1}{(D_1+1)(D_1+1)} \frac{1}{(1-x)^2}$$

$$= \frac{1}{D_1+1}\left[x^{-1}\int x^{1-1}\cdot\frac{1}{(1-x)^2}dx\right]$$

$$= \frac{1}{D_1+1}\left[x^{-1}\int (x-1)^{-2}dx\right]$$

$$= \frac{1}{D_1+1}\left(\frac{1}{x}\cdot\frac{(x-1)^{-2+1}}{-2+1}\right]$$

$$= \frac{1}{D_1+1}\left[\frac{-1}{x(x-1)}\right]$$

A.E.

$$m^2+2m+1=0$$

$$\Rightarrow (m+1)^2=0$$

$$\Rightarrow m=-1,-1$$

$$C.F. = (c_1+c_2 z)e^{-z}$$

$$= [c_1+c_2 \log x]\cdot\frac{1}{x}$$

$\therefore$ Gen. soln. is

$$\underline{y = CF + PI.}$$

$$= x^{-1}\int x^{1-1}\cdot \frac{-1}{x(x-1)}\,dx$$

$$= -\frac{1}{x}\int \frac{1}{x(x-1)}\,dx$$

$$= +\frac{1}{x}\int \left[\frac{1}{x} - \frac{1}{x-1}\right]dx$$

$$= \frac{1}{x}\left[\log x - \log(x-1)\right]$$

$$= \frac{1}{x}\cdot \log \frac{x}{x-1}.$$

Put $a+bx = e^z$

$\Longleftrightarrow z = \log(a+bx)$

$$\frac{dz}{dx} = b \cdot \frac{1}{(a+bx)}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{b}{(a+bx)} \cdot \frac{dy}{dz}$$

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz}$$

$$(a+bx) D y = b D_1 y \quad \Big| \quad D_1 \equiv \frac{d}{dz}$$

$$\rightarrow (a+bx)^n \cdot \frac{d^n y}{dx^n} + A_1 (a+bx)^{n-1} \cdot \frac{d^{n-1} y}{dx^{n-1}}$$

$$+ A_2 (a+bx)^{n-2} \cdot \frac{d^{n-2} y}{dx^{n-2}} + \cdots -$$

$$+ A_n y = Q(x) \quad (1)$$

$A_1, A_2 \text{ --- } A_n$ — constant.

Legendre's linear eqn. of n^{th} order.

Prob: Solve

$$\left[(3x+2)^2 D^2 + 3(3x+2)D - 36\right]y = 3x^2+4x+1 \quad \text{—— (1)}$$

$\longrightarrow$

Put $3x+2 = e^z$

$\Longleftrightarrow \quad z = \log(3x+2)$

(1) $\Rightarrow$

$$\left[3^2 D_1(D_1-1) + 3\cdot 3 D_1 - 36\right]y = \frac{e^{2z}-1}{3} \quad \text{—— (11)}$$

$$(a+bx)^2 \frac{d^2y}{dx^2} = b^2 D_1(D_1-1)y$$

$\vdots \qquad \vdots$

$$(a+bx)^n \frac{d^ny}{dx^n} = b^n \cdot D_1(D_1-1) \cdots \left[D_1-(n-1)\right]y$$

$$= \frac{e^{2z} - \cancel{4e^{z}} + 4 + \cancel{4e^{z}} - 8 + 3}{3}$$

$$= \frac{e^{2z} - 1}{3}.$$

(11) $\Rightarrow$

$$\left[9D_1^2 - \cancel{9D_1} + \cancel{9D_1} - 36\right] y = \frac{e^{2z} - 1}{3}$$

$$3x + 2 = e^{z}$$

$$\Rightarrow x = \frac{e^{z} - 2}{3}.$$

$$3x^2 + 4x + 1$$

$$= 3 \cdot \left[\frac{e^{z} - 2}{3}\right]^2 + 4 \cdot \frac{e^{z} - 2}{3} + 1$$

$$= \frac{1}{3} \cdot \left[e^{2z} - 4e^{z} + 4\right] + \frac{4e^{z} - 8}{3} + 1$$

$$P.I = \frac{1}{3\times 9(D_1^2-4)}\left(e^{2z}-1\right)$$

$$= \frac{1}{27} \cdot \left[\frac{1}{D_1^2-4}e^{2z} - \frac{1}{D_1^2-4}e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{(D_1-2)(D_1+2)}e^{2z} + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{1}{4}\cdot\frac{1}{D_1-2}e^{2z} + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{1}{4}\cdot z e^{2z} + \frac{1}{4}\right]$$

A.E.

$$9m^2 - 36 = 0$$

$$\Rightarrow 9m^2 = 36$$

$$\Rightarrow m^2 = 4$$

$$\Rightarrow m = \pm 2$$

$$C.F. = c_1 e^{2z} + c_2 e^{-2z}$$

$$= c_1 (3x+2)^2 + \frac{c_2}{(3x+2)^2}$$

$$= \frac{1}{108}\left[z \cdot e^{2z} + 1\right]$$

$$= \frac{1}{108}\left[\left(\log(3x+2)\right) \cdot (3x+2)^2 + 1\right]$$

$\therefore$ gen. sol^n. is

$$y = CF + P.I$$

methods to solve this

(I) one integral of C.F. is known.

By changing the dependent var.

(II) changing dependent var. and removal of 1st derivative (Normal form)

Linear Differential eqn. of 2nd order with variable coeff.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q = R \quad \text{--- (1)}$$

where P, Q, R are functions of 'x'.

(iii) changing independent variable.

(IV) Variation of parameter.