Matrices and Linear Algebra

- Algebra of matrices.
- Determinant.
- Adjoint and inverse.
- orthogonal and unitary matrix.
- Rank of a matrix.
- System of linear eqn.

- Eigenvalue and Eigenvector.
- Simalarity of Matrices.

Matrix: — A rectangular array of numbers (real or complex).

Ex:

$$[a_{ij}]_{3\times 2} \longleftarrow A = \begin{array}{cc} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} & \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 4 \end{bmatrix}_{3\times 2} \\ & \begin{array}{cc} C_1 & C_2 \end{array} \end{array} \text{ or } \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 4 \end{pmatrix}_{3\times 2}$$

$a_{ij} = (i, j)$th element

$a_{12} = 2$ $\qquad$ $a_{21} = 2$

$a_{32} = 4$

$A = [a_{ij}]_{m \times n}$

Note $A = [-4]_{1 \times 1} = -4$.

Square matrix : No. of rows = No. of columns.

$A_{m \times n}$ — square — $m = n$.

$\rightarrow A = [a_{ij}]_{n \times n}$

Principal diagonal

a_{ij} — $i = j$

Ex (i) $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}_{2 \times 2}$

(ii) $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 2 & 4 & -4 \end{bmatrix}_{3 \times 3}$

principal diagonal

Null matrix (Zero matrix):

All element should be zero.

Denoted by 'O' $O_{m\times n}$

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$$

$$O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

Row Matrix (Row vector)

(i) $A = \begin{bmatrix} 1 & -1 & 2 & 3 \end{bmatrix}_{1\times 4}$

(ii) $B = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}_{1\times 5}$

Column matrix (column vector).

(i) $A = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}_{4\times 1}$ (ii) $B = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 7 \\ 1 \end{bmatrix}_{6\times 1}$.

upper triangular matrix

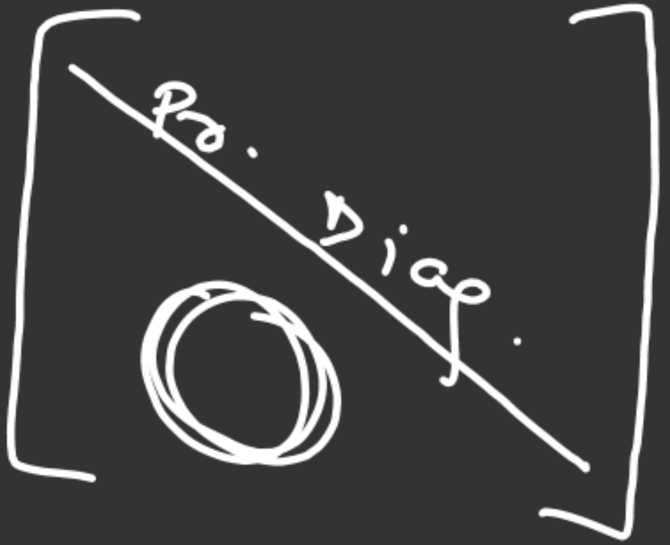
· square matrix ·

Ex: $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & -4 \end{bmatrix}$

$a_{21} = 0$

$a_{31} = 0$

$a_{32} = 0$

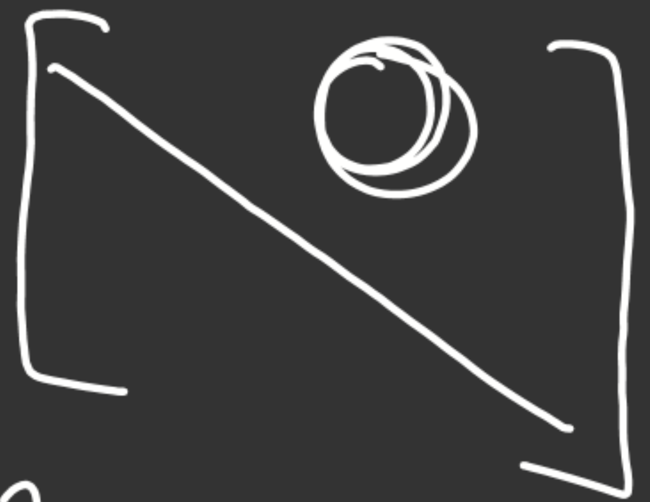
$U =$ 

$\rightarrow$ $U = [u_{ij}]_{n \times n}$ is called upper triangular matrix iff

$$u_{ij} = 0, \quad \forall\, i > j.$$

Lower triangular matrix.

· Square matrix.



$L = [l_{ij}]_{n \times n}$ is called a lower triangular matrix if

$$l_{ij} = 0 \quad \forall\, i < j.$$

Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}_{3 \times 3}$

$$D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}_{3\times 3}$$

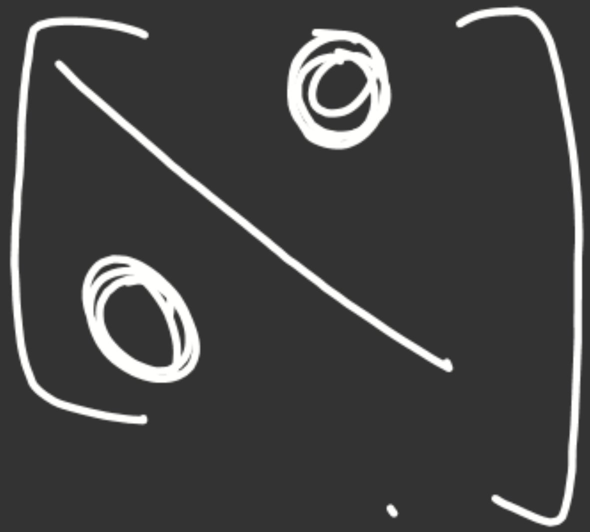
$$= \text{Diag}\,[d_{11}\ d_{22}\ d_{33}]$$

→ Ex: $\text{Diag}\,[1\ \ 0\ \ -1]$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Triangular matrix: UT or LT.

Diagonal matrix: . Square matrix.



$D = [d_{ij}]_{n\times n}$ is called diagonal matrix iff

$d_{ij} = 0 \ \forall\ i \neq j$.

Scalar Matrix:

Ex: (i) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(ii) $B = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

- Diagonal matrix
- All the elements on the principal diagonal must be equal.

unit matrix (Identity matrix)

$$I_n = \begin{bmatrix} 1 & 0 & - & \cdots & 0 \\ 0 & 1 & - & \cdots & 0 \\ \vdots & 0 & - & \cdots & 0 \\ \vdots & \vdots & & \cdots & \\ \vdots & \vdots & - & - & \\ 0 & 0 & - & \cdots & 1 \end{bmatrix}_{n\times n}$$

- Scalar Matrix.
- All the elements on the principal diagonal must be equal to '1'.

Ex (i) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

(ii) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Algebra of matrices

Ex: $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix}_{2\times 3}$

$B = \begin{bmatrix} x & -1 & y-2 \\ 0 & w & t+1 \end{bmatrix}_{2\times 3}$

$A = B$

$x = 1,\quad y-2 = 2 \Rightarrow y = 4$

$w = 2,\quad t+1 = 4 \Rightarrow \underline{t = 3}.$

Equality

$A = [a_{ij}]_{m\times n} \quad B = [b_{ij}]_{m\times n}$

$A = B$

- Same size.
- Corresponding element must be equal.

i.e. $a_{ij} = b_{ij} \quad \forall i, j.$

(Difference)
Sum of two matrices

Ex: $A = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}_{2\times 3}$

$B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}$

$A+B = \begin{bmatrix} 0 & -1 & 4 \\ 2 & 4 & 1 \end{bmatrix}_{2\times 3}$

$A = [a_{ij}]_{m\times n}$ $\quad$ $B = [b_{ij}]_{m\times n}$

$C = A+B$

$\to [c_{ij}]$

- A and B —— same size.
- $c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$

(11) $B = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

$-4B = \begin{bmatrix} -4 & 0 & -12 \\ 4 & -8 & -8 \\ -4 & -4 & +4 \end{bmatrix}$

Scalar multiplication

$A = [a_{ij}]_{m \times n}$

$K \neq 0 \in \mathbb{R}$ (Reals)

$\longrightarrow KA = [K a_{ij}]_{m \times n}$

Ex (1) $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 3 & 4 \end{bmatrix}$

$3A = \begin{bmatrix} 6 & 0 \\ -3 & 3 \\ 9 & 12 \end{bmatrix}.$

Multiplication of matrices

$A_{3\times 2}$ $\quad$ $B_{2\times 4}$

No. of columns = No. of rows

Size of AB $\rightarrow 3\times 4$

AB — defined.
(Conformable for multiplication)

BA — not defined.

No. of col. $\neq$ No. of rows of A

$$= \begin{bmatrix} 7 & 2 \\ -1 & 0 \\ -8 & -1 \end{bmatrix}_{3\times 2}$$

$\longrightarrow$ $A_{3\times 2}$ $\qquad B_{2\times 2}$

No. of col. = No. of rows

AB —— 3×2

Ex:

$$AB = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -1 & 4 \end{bmatrix}_{3\times 2} \begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 2\times 4+(-1)(-1) & 2\times 1+0 \\ -1 & 0 \\ -4-4 & -1+0 \end{bmatrix}_{3\times 2}$$

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -1 \\ 3 & 2 \end{bmatrix}_{2\times 2}$$

<u>Ex:</u>

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}$$

$$B = \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}_{3\times 2}$$

$\underset{2\times 2}{\underline{AB}}$, $\underset{3\times 3}{\underline{BA}}$ — Both defined.

$$\to \quad A = [a_{ij}]_{m\times n} \qquad B = [b_{ij}]_{n\times p}.$$

$C = AB$ — defined. (BA not defined).

$$C = [c_{ij}]_{m\times p}.$$

$$c_{11} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}_{1\times n} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{n1} \end{bmatrix}_{n\times 1}$$

$$= [a_{11}b_{11} + a_{12}b_{21} + a_{13}\cdot b_{31} + \cdots + a_{1n}\cdot b_{n1}]_{1\times 1}$$

$$= a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}\cdot b_{n1}$$

$$= \sum_{k=1}^{n} a_{1k}\cdot b_{k1}$$

$$c_{12} = \sum_{k=1}^{n} a_{1k}\cdot b_{k2}$$

$$c_{21} = \sum_{k=1}^{n} a_{2k}\, b_{k1}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik}\cdot b_{kj}}$$

Note: (i) AB and BA — defined.

Not necessary that

$AB = BA$

(ii) Matrix multiplication

Associative (Always)

$$A(BC) = (AB)C$$

Conformable for multiplication.

Idempotent matrix:

- Square matrix
- $A^2 = A$

$A \times A$.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$$

$i^2 = -1$

(11) $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$B^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Involutory matrix

$\begin{cases} \cdot \text{ Square matrix.} \\ \cdot \ A^2 = I \end{cases}$

Ex (1) $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Nilpotent matrix:

A square matrix A is called a Nilpotent matrix if $\exists$ a +ve integer n s.t.

$$A^n = 0 \text{ (zero matrix)}$$

Note: K is the least (smallest) +ve integer s.t.

$$A^K = 0$$

K is called index.

Ex: $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Index $= 2$.