KGS IAS

Maths Optional

By Dhrub Singh Sir

Prob: Which of the following sets of elements $v = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$?

(I) all v s.t. $a_1 \le 0$.

(II) all v s.t. a_3 is an integer.

(III) all v s.t. $a_2 + 4a_3 = 0$.

(IV) all v s.t. $a_1 + a_2 + \cdots + a_n = K$ (constant)

(1) $W = \{(a_1, a_2, \ldots, a_n) : a_1 \le 0, a_i \in \mathbb{R}\}$

$v = (-1, a_2, \ldots, a_n) \in W$

$-2 \in \mathbb{R}$

$-2v = (\underline{2}, -2a_2, \ldots, -2a_n) \notin W$

$a_1 = 2 > 0$

Not a subspace

(iii) $W = \{ v = (a_1, a_2, \ldots a_n) : a_2 + 4a_3 = 0 \}$

Let $\alpha, \beta \in \mathbb{R}$

Let $u = (a_1, a_2, \ldots a_n) \in W$

$v = (b_1, b_2, \ldots b_n)$

$$\left.\begin{aligned} \therefore\ a_2 + 4a_3 &= 0 \\ b_2 + 4b_3 &= 0 \end{aligned}\right\} \text{—} (1)$$

$\alpha u + \beta v = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3, \ldots)$

(ii) $W = \{ v = (a_1, a_2, \ldots a_n) : a_3 - \text{integer} \}$

Let $v = (a_1, a_2, 2, \ldots a_n) \in W$

Take $a = \frac{1}{3} \in \mathbb{R}$

$$\frac{1}{3} v = \left(\frac{a_1}{3}, \frac{a_2}{3}, \frac{2}{3}, \ldots \frac{a_n}{3} \right) \notin W$$

$\frac{2}{3}$ — Not an int.

Not a subspace.

(iv) Let $\alpha, \beta \in \mathbb{R}$

Let $u, v \in W = \{(a_1, a_2, \ldots a_n) : a_1 + a_2 + \cdots + a_n = k\}$

$u = (a_1, a_2, \ldots a_n)$

$v = (b_1, b_2, \ldots b_n)$ $\in W$

$\therefore a_1 + a_2 + \cdots + a_n = k$

$b_1 + b_2 + \cdots + b_n = k$ — (1)

$\alpha u + \beta v = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \ldots \alpha a_n + \beta b_n)$

$\alpha a_2 + \beta b_2 + 4(\alpha a_3 + \beta b_3)$

$= \alpha(a_2 + 4a_3) + \beta(b_2 + 4b_3)$

$= \alpha \cdot 0 + \beta \cdot 0$

$= 0 + 0 = 0$

$\therefore \alpha u + \beta v \in W$

$\therefore$ W is a subspace of $\mathbb{R}^n$.

$$(\alpha a_1+\beta b_1)+(\alpha a_2+\beta b_2)+\cdots$$
$$+(\alpha a_n+\beta b_n)$$

$$=\alpha(a_1+a_2+\cdots+a_n)$$
$$+\beta(b_1+b_2+\cdots+b_n)$$
$$=\alpha K+\beta K$$
$$=(\alpha+\beta)K$$

If $K=0$, $\alpha u+\beta v\in W$
W is a sub-space.

If $K\neq 0$, $\alpha u+\beta v\notin W$
W is not a sub. sp.

Prob: Show that W is not a sub-space of $\mathbb{R}^3$, where

$$W = \{(a,b,c) : a^2+b^2+c^2 \le 1\} \subseteq \mathbb{R}^3.$$

$u = (1,0,0)$

$v = (0,1,0)$

$\in W$ $\quad 1^2+0^2+0^2 \le 1$

$0^2+1^2+0^2 \le 1$

$u+v = (1,1,0) \notin W$

$1^2+1^2+0^2 > 1$

Closure property doesn't hold in W.

W is not a sub-space of $\mathbb{R}^3$.

(1) Let $a, b \in \mathbb{R}$

Let $f, g \in W$

$\therefore f(3) = 0$

$g(3) = 0$

$$(af + bg)(3)$$

$$= (af)(3) + (bg)(3)$$

$$= a \cdot f(3) + b \cdot g(3)$$

$$= a \cdot 0 + b \cdot 0 = 0$$

$\therefore af + bg \in W$.

Prob: Let V be the real vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$. Which of the following set of functions are sub-spaces of V.

(i) $W = \{f : f(3) = 0\}$.

(ii) $W = \{f : f(7) = f(1)\}$.

(iii) $W = \{f : f(-x) = -f(x)\}$.

$\therefore af + bg \in W$

$\therefore W$ is a subspace.

(iii) let $a, b \in \mathbb{R}$

let $f, g \in W$

$$\therefore \left.\begin{aligned} f(-x) &= -f(x) \\ g(-x) &= -g(x) \end{aligned}\right\} \longrightarrow (1)$$

(ii) $a, b \in \mathbb{R}$

$f, g \in W$

$$\therefore \left.\begin{aligned} f(7) &= f(1) \\ g(7) &= g(1) \end{aligned}\right\} \longrightarrow (1)$$

$$\begin{aligned} (af+bg)(7) &= (af)(7) + (bg)(7) \\ &= a \cdot f(7) + b \cdot g(7) \\ &= af(1) + bg(1) \\ &= (af+bg)(1) \end{aligned}$$

$\therefore af + bg \in W$

$\therefore$ W is a sub-space.

$$(af+bg)(-x)$$
$$= (af)(-x) + (bg)(-x)$$
$$= a \cdot f(-x) + b \cdot g(-x)$$
$$= -a \cdot f(x) - b\, g(x)$$

[using ①]

$$= (-af-bg)(x)$$
$$= -(af+bg)(x)$$

(1) Let $a, b \in \mathbb{R}$

$f, g \in W$

$\therefore$ f and g are continuous functions.

$\therefore$ af and bg are also cts.

$\therefore$ $af + bg$ is also cts.

$\therefore$ $af + bg \in W$.

$\therefore$ W is a subspace.

Prob: In the previous problem consider the following subsets:

(i) $W = \{f : f$ is a continuous function$\}$

(ii) $W = \{f : f$ is a differentiable function$\}$.

Take $u = (2, -1, 1) \in W$

$2 + (-1) = 1 > 0$

Let $a = -1$

$au = -1 \cdot (2, -1, 1)$

$= (-2, 1, -1) \notin W$

$-2 + 1 = -1 < 0$

External composition is not satisfied in W.

$\therefore W$ is not a subspace.

Prob: Show that the set

$$W = \{(a_1, a_2, a_3) : 2a_1 - 3a_2 = a_3\}$$

is a sub-space of $\mathbb{R}^3$.

(Home work)

Prob: Show that the set

$$W = \{(a_1, a_2, a_3) : a_1 + a_2 \geq 0\}$$

is not a subspace of $\mathbb{R}^3$.

(ii) $a, b \in \mathbb{R}$

let $f, g \in W$

$\therefore f'(x) = 0$ $\quad \forall x \in (a,b)$

$g'(x) = 0$ ——— (i)

$$(af + bg)'(x)$$

$$= (af)'(x) + (bg)'(x)$$

$$= a \cdot f'(x) + b \cdot g'(x)$$

$$= a \cdot 0 + b \cdot 0 = 0$$

$\forall x \in (a,b)$

$\mathcal{C}(a,b)$ — vectorspace of real valued continuous functions defined on (a,b).

proof: which of the following set are subspaces of $\mathcal{C}(a,b)$:

(i) $\{f : f(x_0) = 0,\ x_0 \in (a,b)\}$

(ii) $\{f : f'(x) = 0,\ \forall x \in (a,b)\}$

(iii) $\{f : f(x) = x^2 f(x)\}$.

$(af+bg)(x)$

$= (af)(x) + (bg)(x)$

$= a \cdot f(x) + b \cdot g(x)$

$= a \cdot x^2 f(x) + b \cdot x^2 g(x)$

using (1)

$= x^2 \left[a f(x) + b g(x) \right]$

$= x^2 \cdot (af+bg)(x)$

$\therefore af+bg \in W.$

$\therefore$ W is a subspace.

$\therefore af+bg \in W$

$\therefore$ W is a subspace.

(iii) Let $W = \{ f : f(x) = x^2 f(x) \}$

Let $a, b \in \mathbb{R}$

Let $f, g \in W$

$\therefore \left.\begin{array}{l} f(x) = x^2 f(x) \\ g(x) = x^2 g(x) \end{array}\right\} \longrightarrow (1)$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

or

$$\lim_{n\to\infty} a_n = 0$$

or

$$\lim a_n = 0$$

$$\langle 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots \rangle$$

$$\langle a_1, a_2, a_3, a_4, \ldots \rangle$$

$$a_n = \frac{1}{n}$$

$$\langle a_n \rangle \text{ or } \langle \tfrac{1}{n} \rangle$$

$$\langle 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots \tfrac{1}{100}, \ldots \tfrac{1}{10^{10}}, \ldots \rangle$$

$\lim a_n = a \qquad \lim b_n = b$

$\lim (a_n \pm b_n) = a \pm b$

$\times \qquad \times$

$\div \qquad \frac{a}{b}$

$\lim (K a_n)$

$= K \cdot \lim a_n$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$= \sum_{n=1}^{\infty} \frac{1}{n}$

or

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

$\underbrace{\sum a_n = \text{a finite}}_{\text{converge.}}$

$\longrightarrow$ If $\sum a_n$ — converges.

then $\lim a_n = 0$.

Ex: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$

$= \sum \frac{1}{2^n}$ converge

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2^n} = 0$$

(1) Let $a, b \in \mathbb{R}$

Let $u = \langle a_n \rangle$
$v = \langle b_n \rangle$ $\in W$

$\therefore \lim a_n = 0$
$\lim b_n = 0$

$au + bv = a\langle a_n \rangle + b\langle b_n \rangle$
$= \langle a a_n + b b_n \rangle$
$\in W$

As $\lim(a a_n + b b_n)$
$= \lim(a a_n) + \lim(b b_n)$

Prob: Let V be the vector space of all real sequences.

(1) prove that
$W = \{\langle a_n \rangle \in V : \lim a_n = 0\}$
is a sub-space of V.

(11) prove that
$U = \{\langle a_n \rangle \in V : \sum a_n^2 \text{ is finite}\}$
is a subspace of V and is contained in W.

$au+bv$
$= a\langle a_n\rangle + b\langle b_n\rangle \in U$

as $a\underbrace{\sum a_n^2}_{\text{finite}} + b\cdot \underbrace{\sum b_n^2}_{\text{finite}}$ is finite

$\therefore$ U is a sub-space of V

$= a\cdot \lim(a_n) + b\cdot \lim(b_n)$

$= a\cdot 0 + b\cdot 0$

$= 0$

$\therefore$ W is a sub-space of V.

(11) Let $a, b \in \mathbb{R}$

Let $u = \langle a_n\rangle$, $v = \langle b_n\rangle$ $\in U$

$\therefore \sum a_n^2$ and $\sum b_n^2$ are finite.

$\therefore U \subseteq W$.

Let $\langle a_n \rangle \in U$

$\therefore \sum a_n^2$ is finite
(convergent)

$\Rightarrow \lim(a_n^2) = 0$

$\Rightarrow \lim(a_n \cdot a_n) = 0$

$\Rightarrow \lim(a_n) \cdot \lim(a_n) = 0$

$\Rightarrow \lim a_n = 0$

$\Rightarrow \langle a_n \rangle \in W$

Prob: Let V be the vector space of all 2×2 matrices over the field $\mathbb{R}$ of real numbers

Consider

(I) $W_1 = \{A \in V : A^2 = A\}$

(II) $W_2 = \{A \in V : \mathrm{Det}(A) = 0\}$

Show that W_1 and W_2 are not subspace of V.

(1) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A^2 = A$

$\therefore A \in W_1$

Let $2 \in \mathbb{R}$

$2A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$(2A)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

(11) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ $\in W_2$

$\text{Det } A = 0, \ \text{Det } B = 0$

$A + B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\text{Det}(A+B) = 6 \neq 0$

$(2A)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$= 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= 4A \neq 2A$

$\Rightarrow 2A \notin W_1$

$\therefore$ Ext. comp. is not Satisfied in W_1.

$\therefore A+B \notin W_2$

$\therefore$ Closure does not hold in W_2.

$\therefore W_2$ is not a sub-space of V.

Linear combination:

Let $V(F)$ be a vector space. $S = \{v_1, v_2, \ldots v_n\} \subseteq V$

then any vector

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

$$a_i \in F$$

is called a linear combination of the vectors $v_1, v_2, \ldots v_n$.

Ex: $V = \mathbb{R}^2$

$$S = \{\underset{v_1}{(1,0)}, \underset{v_2}{(0,1)}\}$$

$$\underline{2v_1 - 3v_2}$$

$$= 2(1,0) - 3(0,1)$$

$$= (2,0) - (0,3)$$

$$= \underbrace{(2,-3)}_{v}$$

Linear Span: Let $V(F)$ be a vector space. Let $S=\{v_1, v_2, \ldots v_n\} \subseteq V$.

Then collection of all linear combinations of a finite number of elements of S is called the linear span of S, denoted as $L(S)$.

i.e. $L(S)=\{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n : a_i \in F, v_i \in S\}$.

Ex: $V=\mathbb{R}^3$

Let $S=\{e_1=(1,0,0), e_2=(0,1,0), e_3=(0,0,1)\}$

$$L(S)=\{a e_1 + b e_2 + c e_3 : a,b,c \in \mathbb{R}\}$$

$$=\{a(1,0,0)+b(0,1,0)+c(0,0,1) : a,b,c \in \mathbb{R}\}$$

$$=\{(a,b,c) : a,b,c \in \mathbb{R}\}$$

$$=\mathbb{R}^3.$$