KGS IAS

Maths Optional

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HW

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} X = 0$$

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} X = 0$$

$R_3 \to R_3 - R_2$

Rank of coeff matrix $= 2$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda = 2, 2, 1$$

Let X be the eigen vector corr. to $\lambda = 2$

$$(A - 2I)X = 0$$

$3x_2 + 4x_3 = 0$

$-x_3 = 0 \Rightarrow x_3 = 0$

$x_2 = 0$

let $x_1 = K$

$$X = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} = K \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

No of L.I. sol^n $= 3 - 2 = \underline{1}$

Algebraic multiplicity of 2

$\neq$ G M. of 2

Orthogonally similar matrices

Inner product of two vectors:

Let X and Y be two complex n-vectors.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$(X, Y) = \bar{x}_1 \cdot y_1 + \bar{x}_2 \cdot y_2 + \cdots + \bar{x}_n y_n$$

$$= \underline{X^\theta Y}.$$

$$X^\theta Y = [\bar{x}_1, \bar{x}_2 \cdots \bar{x}_n] \begin{bmatrix} y_1 \\ y_2 \\ | \\ y_n \end{bmatrix}$$

$$= \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$$

Norm or length of a vector

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a complex vector.

$$\|X\| = +\sqrt{(X, X)}$$

$$= \sqrt{X^{\theta} X}$$

$$= \sqrt{\bar{x}_1 x_1 + \bar{x}_2 x_2 + \cdots + \bar{x}_n x_n}$$

$$= \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

→ If X and Y are real n-vectors.

$$(X, Y) = X^T Y$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

$$\|X\| = \sqrt{(2)^2 + (1)^2 + (-1)^2}$$
$$= \sqrt{4+1+1} = \sqrt{6}.$$

$$\hat{X} = \frac{X}{\|X\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

<u>unit vector:</u> X

$\underline{\|X\| = 1}$

X - unit vector.

(Normal vector)

<u>Ex:</u> Find a unit vector along $X = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\rightarrow \dfrac{X}{\|X\|} = \hat{X}$

$$x_i^{\theta} \cdot x_j = 0 \quad \forall i \neq j$$

Orthonormal set: A set S of complex n-vectors $x_1, x_2, \ldots, x_k$ are said to be orthonormal if

(I) $x_i^{\theta} \cdot x_j = 0 \quad \forall i \neq j$

(II) $\|x_i\| = 1$

Orthogonal vectors: Let X and Y be two complex n-vectors. X and Y are orthogonal if $X^{\theta} Y = 0$

Orthogonal set: A set S of complex n-vectors $x_1, x_2, \ldots, x_k$ are said to be orthogonal if any two distinct vectors are orthogonal.

Note: If A and B are orthogonally similar then A and B are similar also.

Orthogonally similar matrices

Let A and B be square matrices of order n. Then B is said to be orthogonally similar to A if $\exists$ an orthogonal matrix P s.t.

$$\boxed{P^{-1}AP = B}$$

Result: Any two eigen vectors corresponding to two distinct eigen values of a real symmetric matrix are orthogonal.

Proof: Let A be a real symmetric matrix.

Let λ_1 and λ_2 be two distinct eigenvalue of A.

Results

① Every real symmetric matrix is orthogonally similar to a diagonal matrix with real element.

② A real symmetric matrix of order 'n' has n mutually orthogonal real eigen vectors.

$$= (X_2^T A)\, X_1$$

$$= (X_2^T A^T) \cdot X_1$$

$$[\because A^T = A]$$

$$= (A X_2)^T \cdot X_1$$

$$= (\lambda_2 X_2)^T \cdot X_1$$

$$= \lambda_2 \cdot X_2^T \cdot X_1$$

$$\Rightarrow (\lambda_1 - \lambda_2) X_2^T \cdot X_1 = 0$$

$$\left.\begin{aligned} \because A X_1 &= \lambda_1 X_1 \\ A X_2 &= \lambda_2 X_2 \end{aligned}\right\} \text{———} (1)$$

$\because$ A is a real symmetric matrix.

$\therefore$ λ_1 and λ_2 are real.

Also X_1 and X_2 are real eigen vectors.

$$\lambda_1 X_2^T \cdot X_1 = X_2^T (\lambda_1 X_1)$$

$$= X_2^T (A X_1) \quad (\text{using } (1))$$

eigen vectors corr. to λ.

$\because \lambda_1 \neq \lambda_2$

$\therefore \lambda_1 - \lambda_2 \neq 0$

$\therefore X_2^T \cdot X_1 = 0$

$\therefore X_1$ and X_2 are orthogonal.

Result: If λ occurs exactly p times as an eigen value of a real symmetric matrix A then A has p but not more than p mutually orthogonal real

Prob: Find an orthogonal matrix that will diagonalize the real symmetrix matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

. Also write the resulting diagonal matrix.

$\Rightarrow |A - \lambda I| = 0$

$\lambda^2(\lambda - 14) = 0$

$\lambda = 0, 0, 14$

Let X be the eigen vector corr. to $\lambda = 0$

$(A - 0I)X = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 3R_1$

Let X_2 be the eigen vector corr. to $\lambda=0$ and orthogonal to X_1.

$$X_1^T \cdot X_2 = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \quad 3y - 2z = 0 \quad \text{——— (II)}$$

$y=2, \quad z=3$ satisfy (II)

Now from (I)

$$x+4+9=0$$
$$x=-13.$$

$\therefore$ Rank of coeff. matrix

$=1$

$\therefore$ No. of L.I. soln

$= 3-1 = \underline{2}$

$$x+2y+3z=0 \quad \text{——— (I)}$$

$x=0, \quad y=3$

$6+3z=0$

$3z=-6$

$z=-2$

$$X_1 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -10 & 6 \\ -13 & 2 & 3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} 1 & -5 & 3 \\ -13 & 2 & 3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & -5 & 3 \\ 0 & -63 & 42 \\ 0 & 21 & -14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \to R_2 + 13R_1$
$R_3 \to R_3 - 3R_1$

$$X_2 = \begin{bmatrix} -13 \\ 2 \\ 3 \end{bmatrix}$$

Let X be the eigen vector Corr. to $\lambda = 14$.

$$(A - 14I)X = 0$$

$$\Rightarrow \begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

let $z = k$

$+\cancel{63}^{3}y = +\cancel{42}^{2}k$

$3y = 2k$

$y = \frac{2}{3}k$

suppose $k = 3$

$y = \frac{2}{3} \times 3 = 2$

$x - 5 \times 2 + 3 \times 3 = 0$

$x - 1 = 0$

$x = 1$

$$\begin{bmatrix} 1 & -5 & 3 \\ 0 & -63 & 42 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 + \frac{1}{3}R_2$

Rank of coeff $= 2$

$\therefore$ No. of L.I Sol^n $= 3 - 2 = 1$

$x - 5y + 3z = 0$

$-63y + 42z = 0$

$$\hat{x}_2 = \frac{x_2}{\|x_2\|} = \frac{1}{\sqrt{182}}\begin{bmatrix} -13 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-13}{\sqrt{182}} \\ \frac{2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \end{bmatrix}$$

$$\hat{x}_3 = \frac{x_3}{\|x_3\|} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \quad x_2 = \begin{bmatrix} -13 \\ 2 \\ 3 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Normalize these vectors.

$$\hat{x}_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{13}}\begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{\sqrt{13}} \\ \frac{-2}{\sqrt{13}} \end{bmatrix}$$

$$P = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

and corr. eigen vectors are $(2\ \ 0\ \ 1)$, $(2\ \ 1\ \ 1)$ & $(1\ \ 0\ \ -2)$

clearly they are L.I

Consider

$$P = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

$\because$ col. of P are L.I.

Prob: Let A be a real 3×3 symmetric matrix with eigen values 0, 0, 5. If the corr. eigen vectors are $(2\ \ 0\ \ 1)'$ $(2\ \ 1\ \ 1)$ & $(1\ \ 0\ \ -2)$. Find the matrix A.

Sol^n: Given that A is a real symmetric matrix.

0, 0, 5 are ch. roots.

$$P^{-1} = \begin{bmatrix} 2/5 & -1 & 1/5 \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & -2/5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

$\therefore P$ is non-singular.

Also

$$P^{-1}AP = D$$

$$A = PDP^{-1}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix}^{-1} \quad \text{—} \textcircled{1}$$