KGS IAS

Maths Optional

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Diagonalizable matrix: If a matrix A is similar to a Diagonal matrix then A is said to be Diagonalizable.

In other words, a matrix A is Diagonalizable if $\exists$ an invertible matrix P s.t.

$$P^{-1}AP = D$$

where D is a Diagonal matrix.

Also P is said to diagonalize A or transform A to a diagonal form.

Th: An n-rowed square matrix is diagonalizable iff the matrix possesses n linearly independent eigen vectors.

Proof: Suppose a square matrix $A_{n\times n}$ is diagonalizable.

$\therefore$ A is similar to some Diagonal matrix $D = \mathrm{Diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$.

$\therefore \exists$ some non-singular matrix $P = [X_1, X_2, X_3, \ldots, X_n]$

s.t.

$$P^{-1}AP = D$$

$$\Rightarrow AP = PD$$

$$\Rightarrow A[X_1, X_2, \ldots, X_n]$$

$$= [X_1, X_2, \ldots, X_n]\,\mathrm{Diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$$

$\because$ P is non-singular.

$\therefore$ columns of P i.e.

$X_1, X_2, \ldots X_n$ are

L.I.

$\therefore$ A possesses

n L.I. eigen

vectors.

$$\Rightarrow [AX_1, AX_2, \ldots AX_n]$$

$$= [\lambda_1 X_1, \lambda_2 X_2, \ldots \lambda_n X_n]$$

$$\Rightarrow AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, \ldots$$

$$AX_n = \lambda_n X_n.$$

$\therefore X_1, X_2, \ldots X_n$ are eigen vectors of A coor. to the ch. root $\lambda_1, \lambda_2, \ldots \lambda_n$.

Let $P = [X_1, X_2, \ldots X_n]$

& $D = \text{Diag}[\lambda_1, \lambda_2, \ldots \lambda_n]$

$\because$ columns of P are L.I.

$\therefore$ P is non-singular.

Now $AP = A[X_1, X_2, \ldots X_n]$

$= [AX_1, AX_2, \ldots AX_n]$

$= [\lambda_1 X_1, \lambda_2 X_2, \ldots \lambda_n X_n]$

<u>Conversely</u>: Suppose that the matrix A possesses n L.I. eigen vectors.

Let $X_1, X_2, \ldots X_n$ be the n L.I. eigen vectors of A corr. to the ch. roots $\lambda_1, \lambda_2 \ldots \lambda_n$

$\therefore AX_1 = \lambda_1 X_1,\ AX_2 = \lambda_2 X_2 \ldots AX_n = \lambda_n X_n$

<u>Ex:</u> The matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

has ch. roots $5, 1, 1$ with corresponding eigenvectors $(1,1,1)$, $(2,-1,0)$, $(1,0,-1)$ resp.

Let $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$$AP = [X_1, X_2 \ldots X_n] \cdot \text{Diag.}[\lambda_1, \lambda_2, \ldots \lambda_n]$$

$$= P \cdot D$$

$$\Longrightarrow P^{-1}AP = D$$

$\therefore$ A is similar to D.

$\therefore$ A is diagonalizable.

$$P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \mathrm{Diag}[5, 1, 1]$$

$\Rightarrow (\lambda-1)^3 = 0$

$\Rightarrow \lambda = 1, 1, 1$

Eigen vector corr. to $\lambda = 1$

$(A - I)X = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note: Every square matrix need not be similar to a Diagonal matrix.

Ex: $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ 2 & 2-\lambda & -1 \\ 1 & 2 & -1-\lambda \end{vmatrix}$$

$|A - \lambda I| = 0$

Rank of coeff. matrix

$= 2 <$ No. of var.

$3 - 2 = 1$ — L.I. soln.

Writing eqn. with the help of echelon form

$$x_1 - x_2 + x_3 = 0$$

$$3x_2 - 3x_3 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix}$$

$$= k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= k \cdot X_1$$

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ — L.I}$$

X_1 is the eigen vector corr. to $\lambda = 1$.

A is not Diagonalizable.

$$x_1 - x_2 + x_3 = 0$$

$$3x_2 - 3x_3 = 0$$

$$\Rightarrow \cancel{3}x_2 = \cancel{3}x_3$$

$$\Rightarrow x_2 = x_3$$

Let $x_3 = k$

$$x_2 = k$$

$$x_1 = x_2 - x_3$$

$$= k - k = 0$$

We know that corr. to distinct ch. roots, eigen vectors are L.I.

$\therefore$ A has n L.I. eigen vectors corr. to the distinct ch roots $\lambda_1, \lambda_2, -- \lambda_n$.

$\therefore$ A is diagonalizable.

$\therefore$ A is similar to a Diagonal matrix.

<u>Result</u>: If the ch. roots of an $n \times n$ matrix are all distinct then it is always similar to a Diagonal matrix. (Diagonalizable)

$\boxed{\text{Proof}}$: Let A be an $n \times n$ matrix.

Let $\lambda_1, \lambda_2, -- \lambda_n$ be the distinct ch. roots of A.

A is similar to D

and B is also similar to D.

$\because$ B is similar to D

$\therefore$ D is also similar to B (Symmetric)

So, A is similar to D and D is similar to B

$\therefore$ A is similar to B (Transitivity)

Result: Two $n \times n$ matrices with the same set of n distinct eigenvalues are similar.

Proof: Let A and B be two $n \times n$ matrices.

Let they have same eigen values and i.e. $\lambda_1, \lambda_2 - - \lambda_n$.

Let $D = \text{Diag}[\lambda_1, \lambda_2 - - - \lambda_n]$

$|B-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & -6 & -16 \\ 0 & 17-\lambda & 45 \\ 0 & -6 & -16-\lambda \end{vmatrix} = 0$$

Expanding through C_1,

$$-\lambda\left[(17-\lambda)(-16-\lambda)+45\times 6\right] = 0$$

$$\Rightarrow \lambda\left[-272-17\lambda+16\lambda+\lambda^2+270\right] = 0$$

Prob: Prove that the matrices

$$\begin{bmatrix} -10 & 6 & 3 \\ -26 & 16 & 8 \\ 16 & -10 & -5 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -6 & -16 \\ 0 & 17 & 45 \\ 0 & -6 & -16 \end{bmatrix}$$

are similar.

Soln: Let $A = \begin{bmatrix} -10 & 6 & 3 \\ -26 & 16 & 8 \\ 16 & -10 & -5 \end{bmatrix}$

$B = \begin{bmatrix} 0 & -6 & -16 \\ 0 & 17 & 45 \\ 0 & -6 & -16 \end{bmatrix}$

$$\Rightarrow \lambda\left[\lambda^2 - \lambda - 2\right] = 0$$

$$\Rightarrow \lambda\left[\lambda^2 - 2\lambda + \lambda - 2\right] = 0$$

$$\Rightarrow \lambda(\lambda - 2)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 0, -1, 2.$$

Ch. roots of A —— 0, −1, 2

∴ A and B are similar. (Find it.)

does not change when it is either pre-multiplied or post-multiplied or both by a non-singular matrix.

$\therefore$ Rank of A = Rank of $P^{-1}AP$

= Rank of B

Result: Show that the rank of every matrix similar to A is the same as that of A.

proof: Let B be a matrix similar to A.

$\therefore \exists$ a non-singular matrix P s.t.

$$B = P^{-1}AP.$$

We know that the rank of a matrix

$\therefore$ BA^{-1} and $A^{-1}B$ are similar.

And therefore, they have same eigen values.

Result: Let A and B be n-rowed square matrices and A be non-singular. Show that the matrices $A^{-1}B$ and BA^{-1} have same eigen values.

Proof: $\because$ A is non-singular.

$\therefore$ A^{-1} exists.

$$A^{-1}(\underline{BA^{-1}})A = (A^{-1}B)(A^{-1}A)$$

$$= A^{-1}B \cdot I = \underline{A^{-1}B}$$

Given $U^{\theta} A U = D$

$\Rightarrow U^{-1} A U = D$ [U is non-sing]

$\Rightarrow$ A and D are similar

$\therefore$ A and D have same ch. roots.

Ch. roots of D are $\lambda_1, \lambda_2, \ldots \lambda_n$ $\therefore$ ch. roots of A are $\lambda_1, \lambda_2, \ldots \lambda_n$.

Prob: If U be a unitary matrix s.t. $U^{\theta} A U = \mathrm{Diag}[\lambda_1, \lambda_2, \ldots \lambda_n]$. Show that $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of A.

Soln: Let $D = \mathrm{Diag}[\lambda_1, \lambda_2, \ldots \lambda_n]$

and since U is unitary.

$\therefore U^{\theta} U = I$

$\Rightarrow U^{-1} = U^{\theta}$

$\therefore AB$ and BA are similar.

Prob: If A and B are non-singular matrices of order n, show that the matrices AB and BA are similar.

Sol^n: $\because A$ is non-singular.

$\therefore A^{-1}$ exists.

Now $A^{-1}(AB)A = (A^{-1}A)BA$

$= I \cdot (BA)$

$= BA$

HW **Prob:** A and B are two $n \times n$ matrices with the same set of 'n' distinct eigenvalues. Show that there exists two matrices P and Q (one of them is non-singular) s.t. $A = PQ$ and $B = QP$.