Prob: If A be an $n \times n$ non-singular matrix then $(A^{-1})^{\theta} = (A^{\theta})^{-1}$!

Sol^n: $\because$ A is non-singular.

$\therefore$ A^{-1} exist.

$$A A^{-1} = A^{-1} \cdot A = I$$

$$\Rightarrow (A \cdot A^{-1})^{\theta} = (A^{-1} \cdot A)^{\theta} = I^{\theta}$$

$$\Rightarrow \underline{(A^{-1})^{\theta}} \cdot A^{\theta} = A^{\theta} \cdot (A^{-1})^{\theta} = I$$

Suppose

$$B = (A^{-1})^{\theta}$$

$$B \cdot A^{\theta} = A^{\theta} \cdot B = I$$

$$\therefore (A^{\theta})^{-1} = B$$

$$\Rightarrow (A^{\theta})^{-1} = (A^{-1})^{\theta}$$

Result: $\mathrm{Adj}(A^T) = (\mathrm{Adj}\,A)^T$

proof:

$(i,j)^{th}$ element $(\mathrm{Adj}\,A)^T$

$= (j,i)^{th}$ element of $\mathrm{Adj}\,A$

$=$ cofactor of $(i,j)^{th}$ element of A

$=$ cofactor of $(j,i)^{th}$ element of A^T.

$= (i,j)^{th}$ element of $\mathrm{Adj}(A^T)$

$\therefore (\mathrm{Adj}\,A)^T = \mathrm{Adj}(A^T)$

Prob: If A is a symmetric matrix then show that $\operatorname{Adj} A$ is also symmetric.

Soln: $\because$ A is symmetric.

$$\therefore \; A^T = \underline{A}$$

$$(\operatorname{Adj} A)^T = \operatorname{Adj}(A)^T$$
$$= \operatorname{Adj} A$$

$\therefore$ $\operatorname{Adj} A$ is symmetric.

Prob: If the non-singular matrix A is symmetric then A^{-1} is also symmetric.

Sol^n: $\because$ A is symmetric.

$\therefore A^T = A$.

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

$\Rightarrow A^{-1}$ is symmetric.

Prob: Show that if A is non-singular matrix then $\det(A^{-1}) = (\det A)^{-1}$.

Sol^n: Since A is non-singular

$\therefore |A| \neq 0$

and A^{-1} exists.

$\Rightarrow A \cdot A^{-1} = I$

$\Rightarrow |A \cdot A^{-1}| = |I|$

$\Rightarrow |A| \cdot |A^{-1}| = 1$

$\Rightarrow |A^{-1}| = \dfrac{1}{|A|}$

$\Rightarrow \underline{\mathrm{Det}(A^{-1}) = (\mathrm{Det}\,A)^{-1}}$

Prob: If B is a non-singular matrix then prove that A and $B^{-1}AB$ have the same determinant, A and B being square matrices.

Soln:

$$|B^{-1}AB| = |B^{-1}| \cdot |A| \cdot |B|$$

$$= \frac{1}{\cancel{|B|}} \cdot |A| \cdot \cancel{|B|}$$

$$= |A|.$$

Result: If the product of two non-zero matrices is a zero matrix, show that both of them must be singular matrices.

Proof: Suppose A and B are non-zero square matrices.

$$AB = O \quad (A \neq O, B \neq O)$$

Let, if possible, $|A| \neq 0$

$\Rightarrow A^{-1}$ exist.

$\because AB = O$

$\Rightarrow A^{-1}(AB) = A^{-1} O$

$\Rightarrow (A^{-1}A)B = O$

$\Rightarrow I \cdot B = O$

$\Rightarrow B = O$

which is a contradiction.

$\therefore$ our supposition is wrong.

so $|A| = 0$

Similarly, $|B| = 0$

Ex:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|A| = 0, \quad |B| = 0$$

Result: If $|A| = 0$ then $|\text{adj} A| = 0$

proof: $A \cdot \text{Adj} A = |A| \cdot I_n$

$\because |A| = 0 \quad \therefore A \cdot \text{Adj} A = 0$ — (1)

Case (1) A is a zero matrix.

$\therefore$ Adj A is also a zero matrix

$\therefore |\text{Adj} A| = 0$

Case II A is a non-zero matrix

(1) $\Rightarrow$

$A \cdot \text{Adj} A = 0$

If $\text{Adj} A = 0$ then $|\text{Adj} A| = 0$

If $\text{Adj} A \neq 0$, then also $|\text{adj} A| = 0$ [using the prev. result.]

$A_{n\times n}$

$$\boxed{|KA| = K^n \cdot |A|}$$

Result: If A be an $n\times n$ matrix then prove that $|\text{Adj}A| = |A|^{n-1}$

Proof: we know that

$$A \cdot \text{Adj}A = (\text{Adj}A)\cdot A = |A|\cdot I_n$$

$$\therefore A \cdot \text{Adj}A = |A|\cdot I_n$$

$$\Rightarrow |A \cdot \text{Adj}A| = \text{Det}(|A|\cdot I_n)$$

$$\Rightarrow |A|\cdot|\text{Adj}A| = \underbrace{|A|^n \cdot \text{Det}(I_n)}_{=|A|^n} \quad \text{--- (I)}$$

If $|A| \neq 0$, then dividing both sides of (I) by $|A|$

$$|\text{Adj}A| = |A|^{n-1} \quad \text{--- (II)}$$

If $|A| = 0$, then $|\text{adj}A| = 0$

(II) is also satisfied in this case.

Ex. $|kA_{n\times n}| = k^n \cdot |A|$

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \Rightarrow |A| = 2+6 = 8.$$

$$2A = \begin{bmatrix} 2 & -4 \\ 6 & 4 \end{bmatrix}$$

$$|2A| = \begin{vmatrix} 2 & -4 \\ 6 & 4 \end{vmatrix} = 2^2 \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 2^2 \times |A|$$

$$= 8 + 24$$

$$= 32$$

$$= 2^2 \times 8 = 2^2 \times |A|$$

Prob: If A is a non-singular matrix then show that

$$Adj(Adj\,A) = |A|^{n-2} \cdot A$$

Sol:

$$A \cdot Adj\,A = |A| \cdot I_n \quad \text{--- (1)}$$

Replacing A with $Adj\,A$ in (1)

$$Adj\,A \cdot [Adj(Adj\,A)] = |Adj\,A| \cdot I_n$$

$$\Rightarrow Adj\,A \cdot [Adj(Adj\,A)] = |A|^{n-1} I_n.$$

multiplying both sides by A

$$A[Adj\,A \cdot \{Adj(Adj\,A)\}] = |A|^{n-1} \cdot A \cdot I_n$$

$$(A \cdot Adj\,A) \cdot Adj(Adj\,A) = |A|^{n-1} \cdot A$$

$$(|A| \cdot I_n) \cdot Adj(Adj\,A) = |A|^{n-1} \cdot A$$

$$\Rightarrow I_n \cdot Adj(Adj\,A) = |A|^{n-2} \cdot A$$

$$Adj(Adj\,A) = \text{''}$$

Result: If A and B are two non-singular matrices of the same order then

$$Adj(AB) = Adj\,B \cdot Adj\,A$$

Proof: $\because$ A and B are non-singular

$\therefore$ $|A| \neq 0$ $|B| \neq 0$

Now $|AB| = |A| \cdot |B|$

$\Rightarrow |AB| \neq 0$

$\therefore$ AB is non-singular. —— (i)

$$(AB) \cdot Adj(AB) = |AB| \cdot I \quad \text{—— (ii)}$$

$$AB\,(Adj\,B \cdot Adj\,A)$$

$$= [(AB)\,Adj\,B] \cdot Adj\,A$$

$$= [A \cdot (B \cdot Adj\,B)] \cdot Adj\,A$$

$$= [A \cdot |B| \cdot I] \cdot Adj\,A$$

* pre-multiplying with $(AB)^{-1}$

$$\boxed{(\operatorname{Adj} B)(\operatorname{Adj} A) = \operatorname{Adj}(AB)}$$

$$= |B| \left[A \cdot I \right] \operatorname{Adj} A$$

$$= |B| \cdot \underline{A \cdot \operatorname{Adj} A}$$

$$= |B| \cdot |A| \cdot I_n$$

$$= |A| \cdot |B| \cdot I_n$$

$$(AB)\left[\operatorname{Adj} B \cdot \operatorname{Adj} A\right] = |AB| \cdot I_n \quad \text{(iii)}$$

(ii) & (iii) $\Rightarrow$

* $$(AB)\left[\operatorname{Adj} B \cdot \operatorname{Adj} A\right] = (AB) \operatorname{Adj}(AB)$$

*

$A A^{-1} = A^{-1} \cdot A = I$

$\Rightarrow A \cdot A^T = A^T \cdot A = I$

A is orthogonal iff

$\boxed{A \cdot A^T = A^T \cdot A = I}$

<u>orthogonal matrix :</u> Square matrix.

A — orthogonal iff

$A^T \cdot A = \underline{I}$.

<u>Note</u>: $\because A^T \cdot A = I$

$\Rightarrow |A^T \cdot A| = |I|$

$\Rightarrow |A^T| \cdot |A| = 1$

$\Rightarrow |A| \cdot |A| = 1$

$\Rightarrow (|A|)^2 = 1$

$\Rightarrow |A| = \pm 1$

$\therefore$ A is non-singular.

$\therefore A^{-1}$ exists

As $A^T \cdot A = I$

$\Rightarrow A^{-1} = A^T$ *

Ex ① $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$A^T \cdot A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

$\therefore$ A is orthogonal.

(11) $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

check: $A^T \cdot A = I$

A is orthogonal.

Unitary matrix: A square matrix A is called unitary iff $A^{\theta} \cdot A = I$.

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

$$\Rightarrow (A \cdot A^{-1})^{\theta} = (A^{-1} \cdot A)^{\theta} = I^{\theta}$$

$$\Rightarrow (A^{-1})^{\theta} \cdot A^{\theta} = A^{\theta} (A^{-1})^{\theta} = I$$

$$\Rightarrow (A^{\theta})^{\theta} \cdot A^{\theta} = A^{\theta} (A^{\theta})^{\theta} = I$$

$$\Rightarrow \underline{A \cdot A^{\theta} = A^{\theta} \cdot A = I}$$

$\therefore$ A is unitary iff $A \cdot A^{\theta} = A^{\theta} \cdot A = I$.

Note: $A^{\theta} A = I$

$$\Rightarrow |A^{\theta} \cdot A| = |I|$$

$$\Rightarrow |A^{\theta}| \cdot |A| = 1$$

$$\Rightarrow |\overline{A^{T}}| \cdot |A| = 1$$

$$\Rightarrow \overline{|A|} \cdot |A| = 1$$

$$\Rightarrow |\mathrm{Det}\, A|^2 = 1$$

$$\Rightarrow \mathrm{Det}\, A = \pm 1$$

$$\Rightarrow \mathrm{Det}\, A \neq 0$$

$\therefore$ A is non-singular.

$\exists\, A^{-1}$ exists

As $A^{\theta} \cdot A = I$

$\therefore$ $\underline{A^{-1} = A^{\theta}}$

Ex: $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

Check: A is unitary.

$$A^{\dagger} \cdot A = I$$