Maths Optional

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$$\text{①} \Rightarrow \frac{1}{v + y\frac{dv}{dy}} = \frac{1+e^{v}}{-e^{v}(1-v)}$$

$$\Rightarrow v + y\frac{dv}{dy} = \frac{-e^{v} + v\cdot e^{v}}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^{v}+ve^{v}}{1+e^{v}} - v$$

$$= \frac{-e^{v} + \cancel{ve^{v}} - v - \cancel{ve^{v}}}{1+e^{v}}$$

$$\Rightarrow \frac{1+e^{v}}{-(v+e^{v})}\,dv = \frac{dy}{y}$$

prob:

$$\frac{dy}{dx} = \frac{1+e^{x/y}}{-e^{x/y}\left(1-\frac{x}{y}\right)} \quad \text{— ①}$$

Put $\frac{x}{y} = v$

$$\Rightarrow x = v\cdot y$$

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{v + y\frac{dv}{dy}}$$

$$y\left(\frac{x}{y} + e^{x/y}\right) = c$$

$$\Rightarrow x + y \cdot e^{x/y} = c$$

Integrating both sides

$$-\int \frac{1+e^u}{u+e^u}\,du = \int \frac{dy}{y}$$

$$-\log(u+e^u) = \log y + \text{const.}$$

$$\Rightarrow \log y + \log(u+e^u) = \text{const}$$

$$\Rightarrow \log y(u+e^u) = \text{const}$$

$$\Rightarrow y(u+e^u) = c$$

Non-homogeneous D.E.

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \longrightarrow (A)$$

$$\left(\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\right)$$

Put $x = X + h$

$y = Y + K$.

$$\frac{dy}{dx} = \frac{dY}{dX}$$

(1) $\Rightarrow$

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y + a_1 h + b_1 k + c_1}{a_2 X + b_2 Y + a_2 h + b_2 k + c_2}$$

$$a_1 h + b_1 k + c_1 = 0$$
$$a_2 h + b_2 k + c_2 = 0$$

Calculate h & k.

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y}{a_2 X + b_2 Y}.$$

Prob:

$$\frac{dy}{dx} = \frac{y+x-2}{y-x-4} \quad \text{——} \ (1)$$

put $x = X + h$

$y = Y + K$

Determine h & K s.t

$$h + k - 2 = 0$$
$$-h + k - 4 = 0$$

$$2K = 6$$
$$k = 3$$
$$h = -1$$

(1) becomes,

$$\frac{dY}{dX} = \frac{Y+X}{Y-X} \quad \text{——} \ (11)$$

put $Y = VX$

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

(11) $\Rightarrow$

$$V + X\frac{dV}{dX} = \frac{VX+X}{VX-X} = \frac{V+1}{V-1}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{V+1}{V-1} - V$$

$$x\frac{dv}{dx} = \frac{v+1 - v(v-1)}{v-1}$$

$$= \frac{v+1 - v^2 + v}{v-1}$$

$$= \frac{-v^2+2v+1}{v-1}$$

$$\Rightarrow \frac{v-1}{-v^2+2v+1}\,dv = \frac{dx}{x}$$

$$\int \frac{v-1}{-v^2+2v+1}\,dv = \log x + \text{const}$$

$$-\frac{1}{2}\int \frac{2-2v}{-v^2+2v+1}\,dv = \log x + \text{const.}$$

$$\Rightarrow -\frac{1}{2}\log(-v^2+2v+1) = \log x + \text{const.}$$

$$\Rightarrow \log\frac{1}{\sqrt{-v^2+2v+1}} - \log x = \text{const}$$

$$\Rightarrow \log\frac{1}{x\cdot\sqrt{-v^2+2v+1}} = \text{const.}$$

$$(x+1)^2+2(x+1)(y-3)$$
$$-(y-3)^2=c_1$$

$$\frac{1}{x\sqrt{-v^2+2v+1}}=c$$

$$\Rightarrow \frac{1}{x\sqrt{-\frac{y^2}{x^2}+\frac{2y}{\cancel{x}}+1}}=c$$

$$\Rightarrow \frac{1}{\cancel{x}\cdot\frac{\sqrt{-y^2+2xy+x^2}}{\cancel{x^2}}}=c$$

$$\Rightarrow x^2+2xy-y^2=\frac{1}{c^2}=c_1 \text{ (say)}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 2$$

$$(1) \Rightarrow \quad \frac{dz}{dx} - 2 = \frac{-(z+1)}{2z-1}$$

Ans: $\frac{2}{3}(2x+y) + \frac{1}{3}\log(2x+y-1) = x + C.$

prob: Solve

$$(2x+y+1)\,dx + (4x+2y-1)\,dy = 0$$

$$\Rightarrow -\frac{(2x+y+1)}{(4x+2y-1)} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-[2x+y+1]}{2(2x+y)-1} \quad \text{--- (1)}$$

Put $z = 2x + y$

$$\Rightarrow \frac{dz}{dx} = 2 + \frac{dy}{dx}$$

$$\boxed{\begin{aligned} &U = f(x, y) \\ &dU = \frac{\partial u}{\partial x} dx + \frac{\partial U}{\partial y} \cdot dy \end{aligned}}$$

$$M\,dx + N\,dy = 0$$

$$\Rightarrow \quad dU = 0$$

Integ.

$$\int dU = c$$

$$U = c$$

Exact diff eqn:

$M\,dx + N\,dy = 0$, M and N are fn. of x & y.

This is called exact when $\exists$ a fn. U of x & y s.t.

$$dU = M\,dx + N\,dy$$

i.e. $M\,dx + N\,dy = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$

$d(xy) = 0$

gmf:

$$\boxed{xy = c}$$

Ex: $y\,dx + x\,dy = 0$

$U = xy$

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

$$= \frac{\partial (xy)}{\partial x}dx + \frac{\partial (xy)}{\partial y}\cdot dy$$

$$= y\,dx + x\,dy$$

Prob: Solve

$$(x+2y-2)\,dx + (2x-y+3)\,dy = 0$$

$$M = x+2y-2$$

$$N = 2x-y+3$$

$$\frac{\partial M}{\partial y} = 2 \qquad \frac{\partial N}{\partial x} = 2$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Condition for a diff. eqn. to be exact:

$$M\,dx + N\,dy = 0$$

exact iff $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Soln

$$\int M\,dx + \int N\,dy = c$$

y = const

terms containing fn. of x → ignore

<u>or</u>

$$(x+2y-2)dx+(2x-y+3)dy=0$$

$$(x\,dx-y\,dy)+2(y\,dx+x\,dy)-2\,dx+3\,dy=0$$

$$\Rightarrow \frac{1}{2}(2x\,dx-2y\,dy)+2\,d(xy)-d(2x)+d(3y)=0$$

$$\Rightarrow \frac{1}{2}\left(d(x^2)-d(y^2)\right)+2\,d(xy)-d(2x)+d(3y)=0$$

$\therefore$ Given diff. eqn. is exact.

Soln. is

$$\int(x+2y-2)dx+\int(-y+3)dy=\text{const}$$

y = constant

$$\Rightarrow \frac{x^2}{2}+2xy-2x-\frac{y^2}{2}+3y=\text{const.}$$

$$\Rightarrow x^2+4xy-4x-y^2+6y=c.$$

prob: $y\sin 2x\,dx - (y^2+\cos^2 x+1)dy = 0$

$M = y\sin 2x,\quad N = -y^2-\cos^2 x-1$

$\frac{\partial M}{\partial y} = \sin 2x,\quad \frac{\partial N}{\partial x} = -2\cos x(-\sin x)$

$= 2\sin x\cos x$

$= \sin 2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Exact.

$d\left(\frac{1}{2}x^2 - \frac{1}{2}y^2 + 2xy - 2x + 3y\right) = 0$

Integrate

$\frac{1}{2}x^2 - \frac{1}{2}y^2 + 2xy - 2x + 3y =$ const

$\Rightarrow x^2 - y^2 + 4xy - 4x + 6y = c.$

$M = y\left(1+\frac{1}{x}\right) + \cos y$

$N = \left[x + \log x - x\sin y\right]$

$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$

$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

exact.

$\underline{\text{Sol}^n}$ i

$y\int \sin 2x\, dx - \int (y^2+1)dy = 0$

$\Rightarrow y\left(-\frac{\cos 2x}{2}\right) - \frac{y^3}{3} - y = \text{const.}$

$\Rightarrow 3y\cos 2x + 2y^3 + 6y = C.$

$\underline{\text{Prob}}$: $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx$

$+\left[x + \log x - x\sin y\right]dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial 1}{\partial y} + \frac{\partial e^{x/y}}{\partial y}$$

$$= \frac{\partial e^{x/y}}{\partial (x/y)} \cdot \frac{\partial (\frac{x}{y})}{\partial y}$$

$$= e^{x/y} \cdot x \cdot \frac{\partial y^{(-1)}}{\partial y}$$

$$= -\frac{x}{y^2} \cdot e^{x/y}$$

$$\frac{\partial N}{\partial x} = -\frac{x}{y^2} \cdot e^{x/y} \quad (\text{Do it!})$$

Solⁿ. is

$$y\int dx + y\int \frac{1}{x}\, dx + \cos y \int dx = \text{const.}$$

$$\Rightarrow xy + y\log x + x\cos y = C$$

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prob: $\left(1+e^{\frac{x}{y}}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

$$M = 1 + e^{\frac{x}{y}}$$

$$N = e^{x/y} - \frac{x}{y} \cdot e^{\frac{x}{y}}$$

Integrating factor (I.F.)

If $M\,dx + N\,dy = 0$ is not exact then it can be made exact by multiplying both sides by a fn. $\mu(x, y)$.

Such multiplier is called an Integrating factor.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact.

$$\int \left(1 + e^{\frac{x}{y}}\right) dx = \text{const}$$

$$\Rightarrow x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = \text{const}$$

$$\Rightarrow \boxed{x + y \cdot e^{\frac{x}{y}} = C}$$

$$d\left(\frac{y^2}{x}\right) = \frac{x\,d(y^2) - y^2\,dx}{x^2}$$

$$= \frac{2yx\,dy - y^2\,dx}{x^2}$$

$$d\left(\frac{e^x}{y}\right) = \frac{y\,d(e^x) - e^x\,dy}{y^2}$$

$$= \frac{y \cdot e^x\,dx - e^x\,dy}{y^2}$$

(I) $d(xy) = y\,dx + x\,dy$

(II) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

(III) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(IV) $d\left(\frac{y^2}{x}\right) = \frac{2xy\,dy - y^2\,dx}{x^2}$

(V) $d\left(\frac{e^x}{y}\right) = \frac{y \cdot e^x\,dx - e^x\,dy}{y^2}$

$d(\log xy)$

$\frac{1}{xy} \cdot d(xy)$

$= \frac{y\,dx + x\,dy}{xy}$

or,

$d(\log x + \log y) = d\log x + d\log y$

$= \frac{1}{x}dx + \frac{1}{y}dy$

$= \frac{y\,dx + x\,dy}{xy}$

$d(f(u))$

$= f'(u)\,du$

(VI) $d(\log xy) = \frac{y\,dx + x\,dy}{xy}$

(VII) $d\left(\log \frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy}$

(VIII) $d\left(\log \frac{y}{x}\right) = \frac{x\,dy - y\,dx}{xy}$

(IX) $d\left(\tan^{-1} \frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2 + y^2}$

(X) $d\left(\tan^{-1} \frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$