KGS IAS

Maths Optional

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corresponding to the ch. root λ, then X is also a ch. vector of A and $\frac{1+\lambda}{1-\lambda}$ is the corresponding ch. root.

Prob: Suppose S is an n-rowed real skew-symmetric matrix and I is the unit matrix of order n. Then show that

(i) $I - S$ is non-singular.

(ii) $A = (I+S)(I-S)^{-1}$ is orthogonal.

(iii) $A = (I-S)^{-1}(I+S)$

(iv) If X is a ch. vector of S

(1) $\because$ S is a real skew-symmetric matrix.

$\therefore$ ch. roots of S are either purely imaginary or zero.

$\therefore$ 1 is not a soln. of

$$|S - \lambda I| = 0$$

i.e. $|S - I| \neq 0$ $\quad \left[\because |P| \neq 0 \Rightarrow |-P| \neq 0\right]$

$\Rightarrow |I - S| \neq 0$

$\therefore$ $I - S$ is non-singular

$$= (I+S)^{-1}(I-S)$$

$$\left[\because S^T = -S \quad I^T = I\right]$$

$$A^T \cdot A = (I+S)^{-1}\underline{(I-S)} \cdot \underline{(I+S)} \cdot (I-S)^{-1}$$

$$= \underbrace{(I+S)^{-1}(I+S)}(I-S)\underbrace{(I-S)^{-1}}$$

$$= I \cdot I = I$$

(11) <u>To prove</u> $A = (I+S)(I-S)^{-1}$ is orthogonal.

$$A^T = \left[(I+S)(I-S)^{-1}\right]^T$$

$$= \left[(I-S)^{-1}\right]^T \cdot (I+S)^T$$

$$= \left[(I-S)^T\right]^{-1}(I+S)^T$$

$$= \left[I^T - S^T\right]^{-1}(I^T + S^T)$$

$$(I-S)^{-1}(I-S)(I+S)(I-S)^{-1}$$

$$\underbrace{(I-S)^{-1}(I-S)}_{I} = (I-S)^{-1}(I+S)\underline{(I-S)(I-S)^{-1}}_{I}$$

$$\Rightarrow \underbrace{(I+S)(I-S)^{-1}} = (I-S)^{-1}(I+S)$$

$$\Rightarrow A = (I-S)^{-1}(I+S)$$

$\therefore$ A is orthogonal

$$\left[\because (I+S)(I-S) = (I-S)(I+S)\right]$$

III **To prove:** $A = (I-S)^{-1}(I+S)$

$$\because (I-S)(I+S) = (I+S)(I-S)$$

Pre-multiplying and post multiplying both sides by $(I-S)^{-1}$

$X - SX = X - \lambda X$

$\Rightarrow (I-S)X = (1-\lambda)X$

pre-multiplying both sides by $(I-S)^{-1}$

$X = (1-\lambda)(I-S)^{-1} \cdot X$

$\Rightarrow (I-S)^{-1} X = \frac{1}{1-\lambda} X$ — (ii)

$[\lambda \neq 1]$

(iv) suppose λ is a ch. root of S and X be the corresponding eigen vector.

$SX = \lambda X$

$\Rightarrow X + SX = X + \lambda X$

$\Rightarrow (I+S)X = (1+\lambda)X$ — (i)

$-SX = -\lambda X$

$\therefore \dfrac{1+\lambda}{1-\lambda}$ is the ch. root of A and x is the corr. ch. vector.

Pre-multiplying ① with $(I-S)^{-1}$!

$$(I-S)^{-1}(I+S)x = (I-S)^{-1}(1+\lambda)x$$

$$\Rightarrow Ax = (1+\lambda)(I-S)^{-1}x$$

$$\Rightarrow Ax = (1+\lambda)\cdot\left(\frac{1}{1-\lambda}\right)x$$

using ⑪

$$\Rightarrow Ax = \left(\frac{1+\lambda}{1-\lambda}\right)x$$

Prob: If A is an orthogonal matrix with the property that -1 is not a ch. root, then A is expressible as $(I+S)(I-S)^{-1}$ for some suitable real skew-symmetric S.

Solⁿ: $A = (I+S)(I-S)^{-1}$

Post-multiplying with $I-S$.

$$A(I-S) = (I+S)\frac{(I-S)^{-1}}{}\frac{(I-S)}{}$$

$$\Rightarrow A - AS = (I+S)$$

$$\Rightarrow A - I = AS + S$$

$$\Rightarrow (A-I) = (A+I)S \quad (1)$$

$\because -1$ is not a solⁿ of $|A - \lambda I| = 0$

$\because$ A is orthogonal.

$\therefore$ A is real.

$\therefore$ S is real.

Now we have to show S is skew symmetric.

$$S^T = \left[(A+I)^{-1}(A-I)\right]^T$$

$$= (A-I)^T\left[(A+I)^{-1}\right]^T$$

$\therefore |A-(-1)I| \neq 0$

$\Rightarrow |A+I| \neq 0$

$\therefore$ $A+I$ is non-singular.

$\therefore$ $(A+I)^{-1}$ exists.

Pre-multiplying ① with $(A+I)^{-1}$

$$S = (A+I)^{-1}(A-I)$$

$\therefore S^T = (A^T + I)^{-1}(A^T - I)$

$= [A^T + A^T A]^{-1} [A^T - A^T A]$

$= [A^T(I + A)]^{-1}$ $A^T(I - A)$ [$\because A$ is orth. $\therefore A^T A = I$]

$= (I+A)^{-1} \cdot \underbrace{(A^T)^{-1} \cdot A^T}_{I} (I - A)$

$S^T = (A+I)^{-1} \cdot (I - A)$

$= -(A+I)^{-1}(A - I) = -S.$

$S^T = (A^T - I)\left[(A+I)^T\right]^{-1}$

$= (A^T - I)(A^T + I)^{-1}$ —— (1)

$\therefore (A^T - I)(A^T + I) = (A^T + I)(A^T - I)$

Pre-multiplying and post-multiplying with $(A^T + I)^{-1}$

$(A^T + I)^{-1}(A^T - I) = (A^T - I)(A^T + I)^{-1}$

Prob: If S is a skew-Hermitian matrix, show that the matrix $I-S$ and $I+S$ are both non-singular. Also show that $A = (I+S)(I-S)^{-1}$ is a unitary matrix.

$\rightarrow$ $\because$ S is skew-Hermitian.

$\therefore S^\theta = -S$ —— (i)

$\because$ S is skew-Hermitian

$\therefore$ ch. roots of S are either purely imaginary or zero.

$\therefore$ 1 and -1 are not the soln. of

$$|S - \lambda I| = 0$$

i.e. $|S - I| \neq 0$, $|S + I| \neq 0$

$\Rightarrow |I - S| \neq 0, \; |I + S| \neq 0$

$\therefore$ $I-S$ and $I+S$ are non-singular.

To prove: $A = (I+S)(I-S)^{-1}$ is unitary.

$$A^{\theta} = \left[(I+S)(I-S)^{-1}\right]^{\theta}$$

$$= \left[(I-S)^{-1}\right]^{\theta} \cdot \left[I+S\right]^{\theta}$$

$$= \left[(I-S)^{\theta}\right]^{-1} \cdot (I+S^{\theta})$$

$$= \left[I-S^{\theta}\right]^{-1} \cdot (I-S)$$

$$= \left[I+S\right]^{-1}(I-S)$$

using ①

Now $A^{\theta} \cdot A$

$$= (I+S)^{-1}\underbrace{(I-S)(I+S)}(I-S)^{-1}$$

$$= \underbrace{(I+S)^{-1}(I+S)}\underbrace{(I-S)(I-S)^{-1}}$$

$$= I \cdot I = I.$$

HW prob: If H is a Hermitian matrix, show that

$$A = (I + iH)^{-1}(I - iH)$$ is a unitary matrix. Also show that $A = (I - iH)(I + iH)^{-1}$.

Further show that if λ is a ch. root of H, then $\frac{(1 - i\lambda)}{1 + i\lambda}$ is a ch. root of A.

Result: Similarity of matrices is an equivalence relⁿ. in the set of all $n \times n$ matrices.

Proof:

Reflexive

$$A = I^{-1} A I$$

Similarity of matrices

Defn: Let A and B be two square matrices of order n. Then B is said to be similar to A iff there exist a non-singular matrix P s.t.

$$B = P^{-1} A P$$

$B = (P^{-1})^{-1} \cdot A \, P^{-1}$

$= Q^{-1} A \, Q$ (Let $Q = P^{-1}$)

Q is non-sing.

$\because |P| \neq 0$

$\Rightarrow |P^{-1}| = \frac{1}{|P|} \neq 0$

$\Rightarrow |Q| \neq 0$

$\therefore$ B is similar to A

A is similar to A

$|I| \neq 0$

$\therefore I$ is non-sing.

Symmetric: Suppose A is similar to B.

$\therefore \exists$ some non-singular matrix P s.t.

$$A = P^{-1} B P.$$

$$\Rightarrow B = P A P^{-1}$$

Transitive: Suppose A is similar to B and B is similar to C.

$\therefore$ A is similar to B

$\therefore$ $\exists$ some non-singular matrix P s.t.

$$A = P^{-1} B P$$

Also B is similar to C.

$\therefore$ $\exists$ some non-singular matrix Q s.t.

$$B = Q^{-1} C Q.$$

Now
$$A = P^{-1} B P$$
$$= P^{-1} Q^{-1} C Q P$$
$$= (QP)^{-1} C (QP)$$

$$A = R^{-1} C R$$

$\therefore$ A is similar to C

[let $R = QP$
R is non-sing.
$|R| = |QP|$
$= |Q| \cdot |P| \neq 0$]

$\therefore$ The reln. of similarity is an eq. reln.

Result: Similar matrices have the same determinant.

Proof: Let A and B are similar matrices.

$\therefore \exists$ some non-singular matrix P s.t.

$$A = P^{-1} B P$$

$$\Rightarrow |A| = |P^{-1} B P|$$

$$= |P^{-1}| \cdot |B| \cdot |P|$$

$$= \frac{1}{\cancel{|P|}} \cdot |B| \cdot \cancel{|P|}$$

$$= |B|$$

$$\therefore |A| = |B|$$

$\therefore$ A and B have same determinant.

Th: Similar matrices have the same char. polynomial and hence have the same ch. roots.

proof: Let A and B are similar matrices. $\therefore \exists$ some non-singular matrix P s.t.

$$A = P^{-1} B P$$

So $|A - \lambda I| = |P^{-1} B P - \lambda I|$

$$= |P^{-1} B P - P^{-1}(\lambda I) P|$$

$$= |P^{-1}(B - \lambda I) P|$$

$$= |P^{-1}| \cdot |B - \lambda I| \cdot |P|$$

$$= \frac{1}{\cancel{|P|}} \cdot |B - \lambda I| \cdot \cancel{|P|}$$

$$= |B - \lambda I|$$

$\therefore$ A and B have the same ch. poly.

$\therefore$ have the same ch. roots.

Note: If two matrices of same order have same ch. roots then it is not necessary that they are similar.

Ex: $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{bmatrix}$

have the same ch. roots but are not similar.

Th: If X is a ch. vector of A corr. to the ch. root λ then $P^{-1}X$ is a ch. vector of B corr. to the ch. root λ, where $B = P^{-1}AP$.

Proof: $\because X$ is a ch. vector of A corr. to the ch. root λ.

$$\therefore AX = \lambda X \quad \text{--- (1)}$$

Now $B(P^{-1}X)$

$$= P^{-1}A\underbrace{P \cdot P^{-1}}X$$

$$= P^{-1}A \cdot I \cdot X$$

$$= P^{-1}AX$$

$$= P^{-1}(\lambda X) \quad \text{using (1)}$$

$$= \lambda(P^{-1}X)$$

$$\therefore B(P^{-1}X) = \lambda(P^{-1}X)$$

$\therefore P^{-1}X$ is the eigenvector of B corr. to the ch. root λ.

$\therefore$ Ch. root of D are diagonal element.

$\therefore$ ch. root of A are diagonal element of D.

Th: If the matrix A is similar to a Diagonal matrix D, then the diagonal elements of D are the Ch. root of A.

Proof: $\because$ A is similar to D.

$\therefore$ A and D have same Ch. root.