KGS IAS

Maths Optional

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Row-reduced echelon form.

Prob: Reduce the matrix $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ into Echelon form by E-row operations and hence find its rank.

$$\rightarrow \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$* \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix} \quad R_3 \to R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix} \quad R_3 \to R_3 - 8R_2$$

Rank $= 3$ = No. of non-zero rows.

Prob: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ by reducing it to row-reduced echelon form.

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15 Marks

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 9 & -9 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + 5R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 + 9R_3$$

$\underline{r(A) = 3}$

Let s E-row operations and t E-column operations are applied. Each row(col.) operation is equivalent to pre(post) multiplying with

Theorem: If A be an $m \times n$ matrix of rank 'r' then there exists non-singular matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

Proof: $\because$ Rank$(A) = r$.

$\therefore$ It can be reduced to normal form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ by a finite number of elementary trans.

$\because$ each elementary matrix is non-singular and product of non-singular matrices is also non-singular.

$\therefore P_s P_{s-1} - - P_1 = P$ (say)

and $Q_1 Q_2 - Q_t = Q$ (say)

are non-singular matrices.

$$PAQ = \left[\begin{array}{c|c} I_s & O \\ \hline O & O \end{array}\right]$$

the corresponding elementary matrices.

Let $P_1, P_2, - - - P_s$ be the elementary matrices corresponding to the s E-row operations and let $Q_1, Q_2, - - - Q_t$ be the elementary matrices corresponding to the t column trans.

$$\therefore \underbrace{P_s \cdots P_2 P_1}\, A\, \underbrace{Q_1 Q_2 \cdots Q_t} = \left[\begin{array}{c|c} I_s & O \\ \hline O & O \end{array}\right] \quad \text{--- (1)}$$

28 part

28 part

56 part

52 part Num.

4 part Th.

Paper I

Sec A

1

2

3

4

Sec B

5

6

7

8

Paper II

we know that every row (col.) operation is equivalent to pre (post) multiplying with the corresponding elementary matrices.

Let $P_1, P_2 - - P_s$ be the elementary matrices corr. to E-row op.

Th: Every non-singular matrix is a product of elementary matrices.

proof: Let A be an n-rowed non-singular matrix.

$$\therefore |A| \neq 0$$

$$\therefore \rho(A) = n.$$

So, it can be reduced to I_n by a finite number of elementary trans.

Let $\varphi_1, \varphi_2, \dots \varphi_t$ be the elementary matrices (E-matrices) corr. to the column operations.

$$P_s \dots P_2 P_1 A \varphi_1 \varphi_2 \dots \varphi_t = I_n \quad \text{— (1)}$$

∵ Each E-matrix is non-singular and is therefore invertible.

pre-multiplying successively ~~both~~ sides of (1) with $P_s^{-1}, P_{s-1}^{-1} \dots P_1^{-1}$ and post multiplying both sides of (1) successively

with $\varphi_t^{-1}, \varphi_{t-1}^{-1} \dots \varphi_1^{-1}$.

we set

$$A = P_1^{-1} \dots P_s^{-1} I_n \varphi_t^{-1} \dots \varphi_1^{-1}$$

$$\Rightarrow A = P_1^{-1} P_2^{-1} \dots P_s^{-1} \varphi_t^{-1} \varphi_{t-1}^{-1} \dots \varphi_1^{-1} \quad \text{— (2)}$$

∵ Inverse of E-matrix is also E-matrix.

∴ (2) ⇒

A = Product of E-matrices

definition of E-matrices.

Let E-matrices are $P_1, P_2, --- P_s$.

$$PA = P_1 P_2 P_3 - P_s A \quad (1)$$

Here, A is pre-multiplied with s E-matrices. And we know that each E-row operation is equivalent to pre-multiplying with the

Th: The rank of a matrix does not change by pre-multiplication or post-multiplication with a non-singular matrix.

Proof: Let A be a given matrix.

Let P be a non-singular matrix s.t. PA exists.

$\because$ P is a non-singular matrix.

$\therefore$ P can be written as a product i.e. $r(A\varphi) = r(A)$

where φ is a non-singular matrix.

corresponding E-matrices.

Also, we know that E-operations do not change the rank of a matrix.

$\therefore$ (1) $\Rightarrow$ $\underline{r(PA) = r(A)}$.

Similarly, we can prove that by post-multiplying a matrix by a non-singular matrix will not change the rank of A.

Soln.

$$A_{3\times 3} = I_3 \cdot A \cdot I_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$

Prob: Find two non-singular matrices P and Q such that PAQ is the normal form.

i.e., $PAQ = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right]$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Also find the rank of A.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 - C_1$

$\text{Rank}(A) = 2$

Note: P and Q are not unique