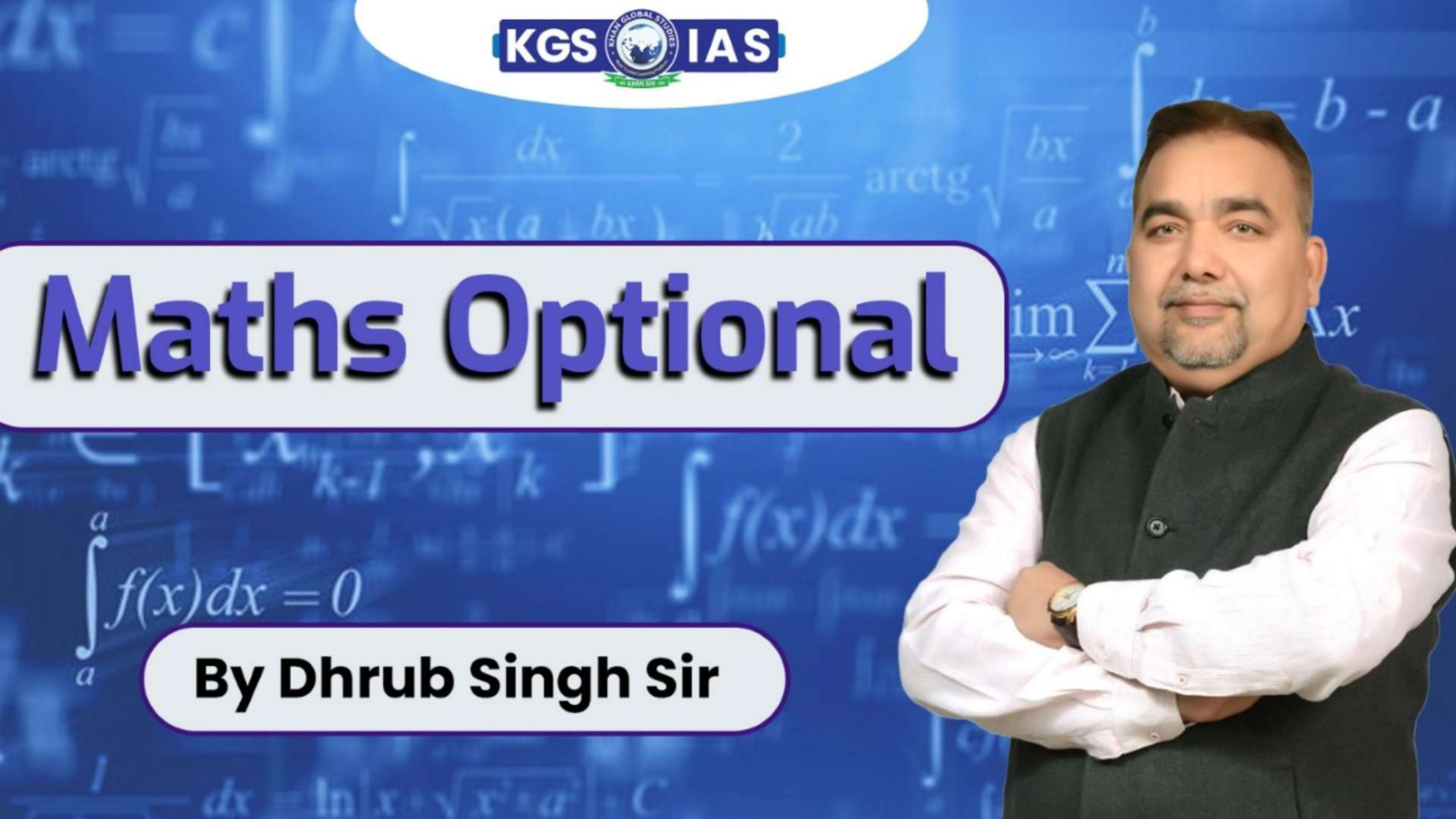
KGS IAS
Maths Optional
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Sol^n: W_1 and W_2 are sub-space of V. (prove it!!) HW

Claim: $V = W_1 + W_2$

clearly, $W_1 + W_2 \subseteq V$,

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}}_{\in W_1} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}}_{\in W_2}$$

Prob: Let V be the vector space of all 2×2 matrices over the field $\mathbb{C}$. Let $W_1 = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in \mathbb{C} \right\}$, $W_2 = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{C} \right\}$

verify that W_1 & W_2 are subspace of V and

$$\dim\left(\frac{W_1 + W_2}{W_2}\right) = \dim\left(\frac{W_1}{W_1 \cap W_2}\right)$$

$\therefore \text{Dim } V = 4$

so $\text{Dim}(W_1 + W_2) = 4$

Basis of $W_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

$\therefore \text{Dim } W_1 = 3$

Basis of $W_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\therefore \text{Dim } W_2 = 2$

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_1 + W_2$

$\therefore V \subseteq W_1 + W_2$

so $V = W_1 + W_2$.

Basis of V

$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$$\dim\left(\frac{W_1}{W_1\cap W_2}\right)=\dim W_1-\dim(W_1\cap W_2)$$
$$=3-1$$
$$=2.$$

$$\therefore \dim\left(\frac{W_1+W_2}{W_2}\right)=\dim\left(\frac{W_1}{W_1\cap W_2}\right)$$

$$W_1\cap W_2=\left\{\begin{bmatrix} x & 0\\ 0 & 0\end{bmatrix}: x\in\mathbb{C}\right\}.$$

Basis of $W_1\cap W_2=\left\{\begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}\right\}$

$$\therefore \operatorname{Dim}(W_1\cap W_2)=1$$

$$\dim\left(\frac{W_1+W_2}{W_2}\right)=\dim(W_1+W_2)-\dim W_2$$
$$=4-2=2$$

Basis of $F[x]$

$= \{1, x, x^2, \cdots\}$

Consider

$W = \{x^2 \cdot f(x) : f(x) \in V\}$

Claim: W is a sub-space of V.

Let $x^2 f(x), x^2 g(x) \in W$

Prob: Give an example of an infinite dimensional vector space $V(F)$ with a subspace W s.t. V/W is finite dimensional.

Sol^n: $V = F[x]$ — vector space of all poly. over the field F.

This is Infinite Dimensional.

$$\frac{V}{W} = \left\{ (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + W : a_0 + a_1 x + \cdots + a_n x^n \in V \right\}$$

$$= \left\{ a_0 + a_1 x + W : a_0, a_1 \in F \right\}$$

$$\left[\because a_2 x^2 + \cdots + a_n x^n = x^2 \underbrace{\left[a_2 + a_3 x + \cdots + a_n x^{n-2} \right]}_{\in F[x] = V} \in W \right.$$

Let $a, b \in F$.

Now $a \cdot (x^2 f(x)) + b \cdot (x^2 g(x))$

$$= x^2 \left[a f(x) + b g(x) \right] \in W$$

As $a f(x) + b g(x) \in V$

$\therefore W$ is a subspace of V

Basis of $\frac{V}{W} = \{1+W,\ x+W\}$

$\therefore \text{Dim} \frac{V}{W} = 2$

V/W — FDVS

Vector Space homomorphism

Let U and V be the vector space over the field F.
The function $f : U \to V$ is called a homomorphism (Linear transformation)

if (i) $f(u+v) = f(u) + f(v)$

(ii) $f(\alpha u) = \alpha . f(u)$ $\quad \forall u, v \in U$, $\alpha \in F$.

or

if $f(au+bv) = a f(u) + b f(v)$

$\forall u, v \in U$

$\forall a, b \in F$

(3) If $f : U \to U$ then f is called a linear operator on U.

(4) Suppose $T : U \to V$ be a linear transformation (L.T.). Then for any $a_i \in F$, $v_i \in U$

$$T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n).$$

Note (1) If f is onto fn. then V is called homomorphic image of U.

(2) If f is 1-1 & onto fn. then f is called isomorphism. V is called isomorphic to U.

$$U \cong V.$$

This L.T. is called zero transformation.
Denoted by 'O'.

⑥ Identity operator:

Let $V(F)$ be a vector space.
$I : V \to V$ be defined as:
$$I(v) = v, \ \forall v \in V$$

⑤ Let $T : U \to V$ be defined as.
$$T(u) = 0 \quad \forall u \in U.$$

T is a L.T.

$$T(au_1 + bu_2) \qquad u_1, u_2 \in U \quad a, b \in F$$
$$= 0$$
$$= a \cdot 0 + b \cdot 0$$
$$= a \cdot T(u_1) + b \cdot T(u_2)$$

(7) Negative of a L.T.

Let U and V are vector spaces over the field F.

Let $T: U \to V$ be a L.T.

Then the function $(-T): U \to V$ defined as: $(-T)(u) = -T(u) \quad \forall u \in U$

I is a L.T.

Let $a, b \in F$, $u, v \in V$

$$I(au+bv)$$

$$= au + bv$$

$$= a \cdot I(u) + b \cdot I(v)$$

Ex: Show that the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(\alpha, \beta) = (\alpha, 0)$ is a L.T.

Sol^n: Let $a, b \in \mathbb{R}$

Let $u = (\alpha_1, \beta_1)$
$v = (\alpha_2, \beta_2)$ $\in \mathbb{R}^2$

$$T(au + bv) = T\left[a(\alpha_1, \beta_1) + b(\alpha_2, \beta_2)\right]$$

$-T$ is a L.T.

Let $a, b \in F$, $u, v \in U$

$$(-T)(au + bv)$$
$$= -T(au + bv)$$
$$= -\left[a\,T(u) + b \cdot T(v)\right]$$
$$= a(-T(u)) + b(-T(v)) \quad \because T \text{ is L.T.}$$

$\therefore -T$ is a L.T.

prob: Show that the mapping $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2) = (x_1 + x_2,\ x_1 - x_2,\ x_2)$$

is a L.T.

Sol^n: Let $a, b \in \mathbb{R}$

Let $u = (x_1, x_2)$ $v = (y_1, y_2) \in \mathbb{R}^2$

$$au + bv = (ax_1 + by_1,\ ax_2 + by_2)$$

$$= T[a\alpha_1 + b\alpha_2,\ a\beta_1 + b\beta_2]$$

$$= (a\alpha_1 + b\alpha_2,\ 0)$$

$$= (a\alpha_1,\ 0) + (b\alpha_2,\ 0)$$

$$= a(\alpha_1,\ 0) + b(\alpha_2,\ 0)$$

$$= a \cdot T(\alpha_1, \beta_1) + b \cdot T(\alpha_2, \beta_2)$$

$$= a \cdot T(u) + b \cdot T(v)$$

$\therefore$ T is a L.T.

$= (ax_1 + ax_2,\ ax_1 - ax_2,\ ax_2)$

$+ (by_1 + by_2,\ by_1 - by_2,\ by_2)$

$= a(x_1 + x_2,\ x_1 - x_2,\ x_2)$

$+ b(y_1 + y_2,\ y_1 - y_2,\ y_2)$

$= a \cdot T(x_1, x_2) + b \cdot T(y_1, y_2)$

$= a \cdot T(u) + b \cdot T(v)$

$\therefore$ T is L.T.

$T(au + bv)$

$= T(ax_1 + by_1,\ ax_2 + by_2)$

$= (ax_1 + by_1 + ax_2 + by_2,$

$ax_1 + by_1 - ax_2 - by_2,\ ax_2 + by_2)$

$= (ax_1 + ax_2 + by_1 + by_2,$

$ax_1 - ax_2 + by_1 - by_2,\ ax_2 + by_2)$

$$T(\alpha Z) = T(2)$$
$$= T(2 + 0 \cdot i)$$
$$= 2.$$

$$\alpha \cdot T(Z) = (1+i) \cdot T(1-i)$$
$$= (1+i) \cdot 1$$
$$= 1+i$$

$$\therefore T(\alpha Z) \neq \alpha \cdot T(Z).$$

Prob: Is the mapping $T: \mathbb{C} \to \mathbb{C}$ defined by $T(x+iy) = x$, a linear trans.? ($\mathbb{C}$ is the vector space over itself.)

Sol^n: Let $\alpha = (1+i)$

$$Z = (1-i)$$

$$\alpha Z = (1+i)(1-i)$$
$$= 1 - i^2 = 1 - (-1)$$
$$= 2$$

$T(\alpha u)$

$= T[2(1,0)]$

$= T[2,0]$

$= 2^2 + 0^2$

$= 4$

$\therefore T(\alpha u) \neq \alpha T(u)$

T is not a L.T.

Prob: Show that the mapping $T: \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x_1, x_2) = x_1^2 + x_2^2$ is not a L.T.

Soln: Take $\alpha = 2$

$u = (1,0)$

$\alpha \cdot T(u) = 2 \cdot T(1,0)$

$= 2 \cdot (1^2 + 0^2)$

$= 2$

Now
$T(x,y) = T[x(1,0) + y(0,1)]$

$= x \cdot T(1,0) + y \cdot T(0,1)$

[As T is L.T.]

$= x \cdot (1,1) + y \cdot (-1,1)$

$= (x, x) + (-y, y)$

$$\boxed{T(x,y) = (x-y,\ x+y)}$$

<u>Prob:</u> Describe explicitly the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(1,0) = (1,1)$, $T(0,1) = (-1,1)$.

<u>Sol^n:</u> $\{(1,0), (0,1)\}$ is a basis of $\mathbb{R}^2$.

Let $(x,y) \in \mathbb{R}^2$

$(x,y) = x(1,0) + y(0,1)$

$\Rightarrow (a, 2a+b) = (0,0)$

$\Rightarrow a=0, \quad 2a+b=0$

$b=0$

Let $(x,y) \in \mathbb{R}^2$.

Let $(x,y) = a(1,2) + b(0,1)$

$\Rightarrow (x,y) = (a, 2a+b)$

$a = x, \quad 2a+b = y$

$b = y - 2x$

Prob: Describe explicitly the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(1,2) = (2,3)$, $T(0,1) = (1,1)$.

Sol^n: Claim: $\{(1,2), (0,1)\}$ is a basis of $\mathbb{R}^2$.

$a(1,2) + b(0,1) = (0,0)$

prob: Give an example of a linear operator T on $\mathbb{R}^2$ s.t. $T^2 = 0$ but $T \neq 0$.

Sol^n: Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x, y) = (y, 0)$$

claim: T is L.T.

Prove it! (HW)

Now $T(x, y) = T\left[a(1,2) + b\cdot(0,1)\right]$

$= a\cdot T(1,2) + b\cdot T(0,1)$

[As T is L.T.]

$= a\cdot(2,3) + b\cdot(1,1)$

$= (2a+b,\ 3a+b)$

$= (\cancel{2x} + y - \cancel{2x},\ 3x + y - 2x)$

$= (y,\ x+y)$

$T(1,2) = (2,0) \neq (0,0)$

$\therefore T \neq 0 \rightarrow L \cdot T.$

$T(x,y) = (y,0)$

$T^2(x,y) = T\big(T(x,y)\big)$

$= T(y,0)$

$= (0,0)$

$\therefore T^2(x,y) = (0,0)$

$\forall (x,y) \in \mathbb{R}^2$

$\therefore \underline{T^2 = 0}$

$$I(x)=\int_0^x f(x)\,dx \quad \text{HW}$$

are L.T.

soln.

(I) Let $a, b \in \mathbb{R}$

$f(x), g(x) \in V$

$$D(af+bg)$$

$$= \frac{d}{dx}(af+bg)$$

$$= a\frac{df}{dx}+b\frac{dg}{dx}.$$

$$= a\cdot D(f)+b\cdot D(g).$$

Prob: Let V be the vector space of polynomials in the variable 'x' over the field $\mathbb{R}$. Let $f(x) \in V$.

Show that

(I) $D: V \to V$ defined by

$$D f(x) = \frac{d}{dx} f(x).$$

(II) $I: V \to V$ defined by

Sol^n:

Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in P_n(\mathbb{R})$

$$\left[1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!}\right](f(x))$$

$$= \cdots\cdots$$

$$= f(x+1)$$

$$= T(f(x))$$

(Remaining steps. HW)

Prob: Let $P_n(\mathbb{R})$ be the vector space of all polynomials of degree 'n' over a field $\mathbb{R}$. If a linear operator T on $P_n(\mathbb{R})$ is s.t.

$$T(f(x)) = f(x+1), \quad f(x) \in P_n(\mathbb{R})$$

prove that

$$T = 1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!}.$$