KGS IAS

Maths Optional

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Linear Diff. eqn. with constant Coefficients.

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = \phi. \quad \text{—— (1)}$$

$a_1, a_2, \ldots a_n$ — Constants.

ϕ — fn. of 'x'.

$D \equiv \frac{d}{dx} \quad D^2 \equiv \frac{d^2}{dx^2} \cdots$

(1) $\Rightarrow$

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_n y = \phi$$

i.e. $\underbrace{(D^n + a_1 D^{n-1} + \cdots + a_n)}_{f(D)} y = \phi$

$$f(D)\, y = \phi \quad \text{—— (11)}$$

Where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n$.

$y = g(x)$ is a particular sol^n. of (ii)
(particular Integral)

$y = f(x) + g(x)$ — general sol^n. of (ii)

$f(x) \to$ C.F. $\quad g(x) \to$ P.I.

$\to$ $f(D)\,y = 0$ — C.F.

$$P.I. = \frac{1}{f(D)}\,\varphi \text{ — P.I.}$$

If $\varphi = 0$,

(ii) — homogeneous eqn.

$f(D)\,y = 0$ — (iii)

General sol^n. of (ii)

$y = f(x)$ is a general sol^n. of (iii) — complementary fn. (C.f.)

Case I : Roots are real and distinct.

$m_1, m_2, - \text{———} m_n$

$(D-m_1)y=0$

$\frac{dy}{dx} - m_1 y = 0$

$I.F. = e^{\int -m_1 dx}$
$= e^{-m_1 x}$

$y \cdot e^{-m_1 x} = \int 0 \cdot e^{-m_1 x} dx + c$

$\boxed{y = c e^{m_1 x}}$

How to find C.F.

$f(D)y=0$

$[D^n + a_1 D^{n-1} + - - + a_{n-1} D + a_n] y = 0$ — (A)

Auxilliary eqn.

$m^n + a_1 m^{n-1} + - - - + a_{n-1} m + a_n = 0$

i.e. $f(m)=0$ — (B)

$m_1 = m_2$, $m_3, m_4, \ldots - m_n$

General soln. of (A)

$$y = (c_1 + c_2 \cdot x) e^{m_1 x} + c_3 e^{m_3 x}$$
$$+ c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

$$(D - m_1)^2 y = 0$$

$$\Rightarrow (D - m_1) \underbrace{(D - m_1) y}_{V} = 0$$

$$\Rightarrow (D - m_1) V = 0$$

Soln. of (A),

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

C.F.

$c_1, c_2, \ldots - c_n$ are arbitrary constants.

Case (ii) when two roots of eqn. (B) are equal and remaining roots different.

$$y \cdot e^{-m_1 x} = c_1 x + c_2$$

$$\Rightarrow y = (c_1 x + c_2) e^{m_1 x}$$

Case (iii) When three roots of eqn. B are equal, remaining diff.

$$m_1 = m_2 = m_3 \,,\; m_4, m_5 - m_n$$

$$y = (c_1 + c_2 x + c_3 x^2) e^{+m_1 x} + c_4 e^{+m_4 x} + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}$$

$$V = c_1 e^{m_1 x}$$

$$(D - m_1) y = V$$

$$\Rightarrow (D - m_1) y = c_1 e^{m_1 x}$$

$$\Rightarrow \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

$$I.F. = e^{\int -m_1 dx} = e^{-m_1 x}$$

$$y \cdot e^{-m_1 x} = c_1 \int e^{-m_1 x} \cdot e^{m_1 x} dx + c_2$$

$$y = \underbrace{c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$= e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right] + \;''$$

$$= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right] +$$

$$= e^{\alpha x}\left[\underbrace{(c_1+c_2)}_{A}\cos\beta x + \underbrace{i(c_1-c_2)}_{B}\sin\beta x\right] \;'' + \cdots$$

$$= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right] + \;''$$

r roots are equal

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_r x^{r-1}) \cdot e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}.$$

Case IV

$m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$,

$\underbrace{m_3, m_4 \cdots, m_n}$

Real & Distinct.

Prob: Find C.F. of $(D^2-3D+2)y=0$

Sol^n: A.E.

$$m^2-3m+2=0$$

$$m^2-2m-m+2=0$$

$$\Rightarrow m(m-2)-1(m-2)=0$$

$$\Rightarrow (m-1)(m-2)=0$$

$$m=1,2$$

$$y=c_1e^{x}+c_2e^{2x}$$

Case V

$\alpha+i\beta, \alpha-i\beta, \alpha+i\beta, \alpha-i\beta,$ $\underbrace{m_5, m_6 - m_n}_{\text{Real \& Dist}}$

$$y=e^{\alpha x}\left[(A+Bx)\cos\beta x+(C+Dx)\sin\beta x\right]$$

$$+c_5e^{m_5x}+\dots+c_ne^{m_nx}.$$

Prob: Solve $(D^4+D^2+1)y=0$

A.E.

$$m^4+m^2+1=0$$

$$(m^2+1)^2 = m^4+2m^2+1$$

$$\Rightarrow (m^2+1)^2 - m^2 = m^4+m^2+1$$

$$\Rightarrow (m^2+1-m)(m^2+1+m) = m^4+m^2+1$$

$$m^4+m^2+1=0$$

$$\Rightarrow (m^2+m+1)(m^2-m+1)=0$$

$m^2+m+1=0$ or $m^2-m+1=0$

$$m=\frac{-1\pm\sqrt{1-4}}{2}, \quad m=\frac{1\pm\sqrt{1-4}}{2}$$

$$=\frac{-1\pm\sqrt{3}\,i}{2}, \quad m=\frac{1\pm\sqrt{3}\,i}{2}$$

$$=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i, \quad m=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

$m^4-2m^3+2m^2-2m+1=0$

$(m-1)\left[m^3-m^2+m-1\right]=0$

$(m-1)(m-1)\left[m^2+1\right]=0$

$\Rightarrow m=1,1,\quad m^2=-1$

i.e. $m=\pm i$

$y=(c_1+c_2x)e^{x}+c_3\cos x+c_4\sin x$

$y=e^{-\frac{1}{2}x}\left[c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right]$

$+e^{\frac{1}{2}x}\left[c_3\cos\frac{\sqrt{3}}{2}x+c_4\sin\frac{\sqrt{3}}{2}x\right]$

prob: $(D^4-2D^3+2D^2-2D+1)y=0$

A.E.

$m^4-2m^3+2m^2-2m+1=0$

$m=1$ $\quad 1-2+2-2+1$

$=0$

D — Differential operator

$\frac{1}{D}$ — Integral operator.

$$D x^2 = \frac{d}{dx}(x^2) = 2x$$

$$\frac{1}{D}(2x) = \int 2x\,dx$$

$$= x^2$$

$\frac{1}{D}$ — inverse operator w.r.t D.

Particular Integral (P.I.)

$$f(D)\,y = \varphi.$$

$$P.I. = \frac{1}{f(D)}\varphi$$

$$f(D)\left[\frac{1}{f(D)}\varphi\right] = \varphi$$

$\frac{1}{f(D)}$ — inverse operator w.r.t. $f(D)$

$$y \cdot e^{-ax} = \int e^{-ax} \cdot X \, dx$$

$$\Rightarrow \boxed{y = e^{ax} \int e^{-ax} \cdot X \, dx}$$

$$\boxed{\frac{1}{D+a} X = e^{-ax} \int X \cdot e^{ax} \, dx}$$

A useful relⁿ:

$$\frac{1}{D-a} X = e^{ax} \int X \cdot e^{-ax} \, dx$$

Let $\frac{1}{D-a} X = y$

$$\Rightarrow X = (D-a) y$$

$$\Rightarrow \frac{dy}{dx} - ay = X$$

$$IF = e^{\int -a \, dx}$$
$$= e^{-ax}$$

$f(D)\,y = \varphi$

$$f(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} y + a_n$$

$$D e^{ax} = \frac{d}{dx} e^{ax} = a e^{ax}$$

$$D^2 e^{ax} = \frac{d^2}{dx^2}(a e^{ax}) = a^2 e^{ax}$$

$\vdots$

$$D^n e^{ax} = a^n e^{ax}$$

$$f(D)[e^{ax}] = [a^n e^{ax} + a_1 a^{n-1} e^{ax} + \cdots + a_{n-1} e^{ax} + a_n]$$

Methods for finding P.I.

(1) $\varphi = e^{ax}$

Case (1) $f(a) \neq 0$

$$\frac{1}{f(D)}(e^{ax}) = \frac{e^{ax}}{f(a)}$$

Ex: $f(D) = (D-1)(D-2)$

$$\frac{1}{f(D)} e^{3x} = \frac{e^{3x}}{(3-1)(3-2)} = \frac{e^{3x}}{2}$$

Case II $f(a) = 0$

$$\underline{\underline{f(D) = (D-a)^r \phi(D)}}$$

s.t. $\phi(a) \neq 0$.

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^r \phi(D)} e^{ax}$$

$$= \frac{x^r}{\lfloor r} \cdot \frac{e^{ax}}{\phi(a)}$$

$$f(D)[e^{ax}] = f(a) \cdot e^{ax}.$$

$$\Rightarrow \frac{1}{f(D)} \left\{ f(D) e^{ax} \right\} = \frac{1}{f(D)} f(a) \cdot e^{ax}$$

$$\Rightarrow e^{ax} = \underline{f(a)} \cdot \frac{1}{f(D)} e^{ax}$$

$$\Rightarrow \boxed{\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}}$$

$$= e^{ax}\int e^{-ax} \cdot x e^{ax}\,dx$$

$$= e^{ax}\int x\,dx$$

$$= e^{ax} \cdot \frac{x^2}{\lfloor 2}$$

$$\frac{1}{(D-a)^3} e^{ax} = \frac{x^3}{\lfloor 3} \cdot e^{ax}$$

$$\vdots \qquad \vdots$$

$$\frac{1}{(D-a)} e^{ax}$$

$$= e^{ax}\int e^{-ax} \cdot e^{ax}\,dx$$

$$= e^{ax}\int dx$$

$$= x e^{ax}$$

$$\frac{1}{(D-a)^2} e^{ax} = \frac{1}{D-a}\left(\frac{1}{D-a} e^{ax}\right)$$

$$= \frac{1}{D-a} x e^{ax}$$

Prob: Solve

$$(D^2+D+1)y = e^{-x}$$

A.E.

$$m^2+m+1=0$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\,i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$C.F. = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$P.I. = \frac{1}{D^2+D+1} e^{-x}$$

$$= \frac{e^{-x}}{(-1)^2-1+1} = e^{-x}$$

General solⁿ is

$$y = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{-x}.$$

$$P.I. = \frac{1}{D^3+3D^2+3D+1} e^{-x}$$

$$= \frac{1}{(D+1)^3} e^{-x}$$

$$= \frac{x^3}{\lfloor 3} e^{-x}$$

General solⁿ. is

$$y = (c_1 + c_2 x + c_3 x^2) e^{-x} + \frac{x^3}{\lfloor 3} e^{-x}$$

Prob: Solve

$$(D^3+3D^2+3D+1)y = e^{-x}.$$

Solⁿ:

A.E. is

$$m^3+3m^2+3m+1=0$$

$$\Rightarrow (m+1)^3 = 0$$

$$\Rightarrow m = -1, -1, -1$$

$$C.F. = (c_1 + c_2 x + c_3 x^2) e^{-x}.$$

prob: Solve $(4D^2-1)y = e^{\frac{x}{2}}$.

A.E is

$$4m^2-1=0$$

$$4m^2=1$$

$$m^2=\frac{1}{4}$$

$$m=\pm\frac{1}{2}$$

$$C.F = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}$$

$$PI = \frac{1}{4D^2-1} e^{\frac{1}{2}\cdot x}$$

$$= \frac{1}{4\left(D^2-\frac{1}{4}\right)} e^{\frac{x}{2}}$$

$$= \frac{1}{4\left(D-\frac{1}{2}\right)\left(D+\frac{1}{2}\right)} \cdot e^{\frac{1}{2}x}$$

$$= \frac{1}{4\left(D-\frac{1}{2}\right)\cdot\left(\frac{1}{2}+\frac{1}{2}\right)} \cdot e^{\frac{x}{2}}$$

$$= \frac{1}{4\left(D-\frac{1}{2}\right)} e^{\frac{1}{2}x} = \frac{1}{4}\cdot\frac{x}{1!} e^{\frac{x}{2}}$$

General soln. is

$$y = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x} + \frac{x \cdot e^{x/2}}{4}.$$

Prob: Solve

$$(D^3 - 5D^2 + 7D - 3)y = e^{2x} \cosh x$$

HW