KGS IAS

Maths Optional

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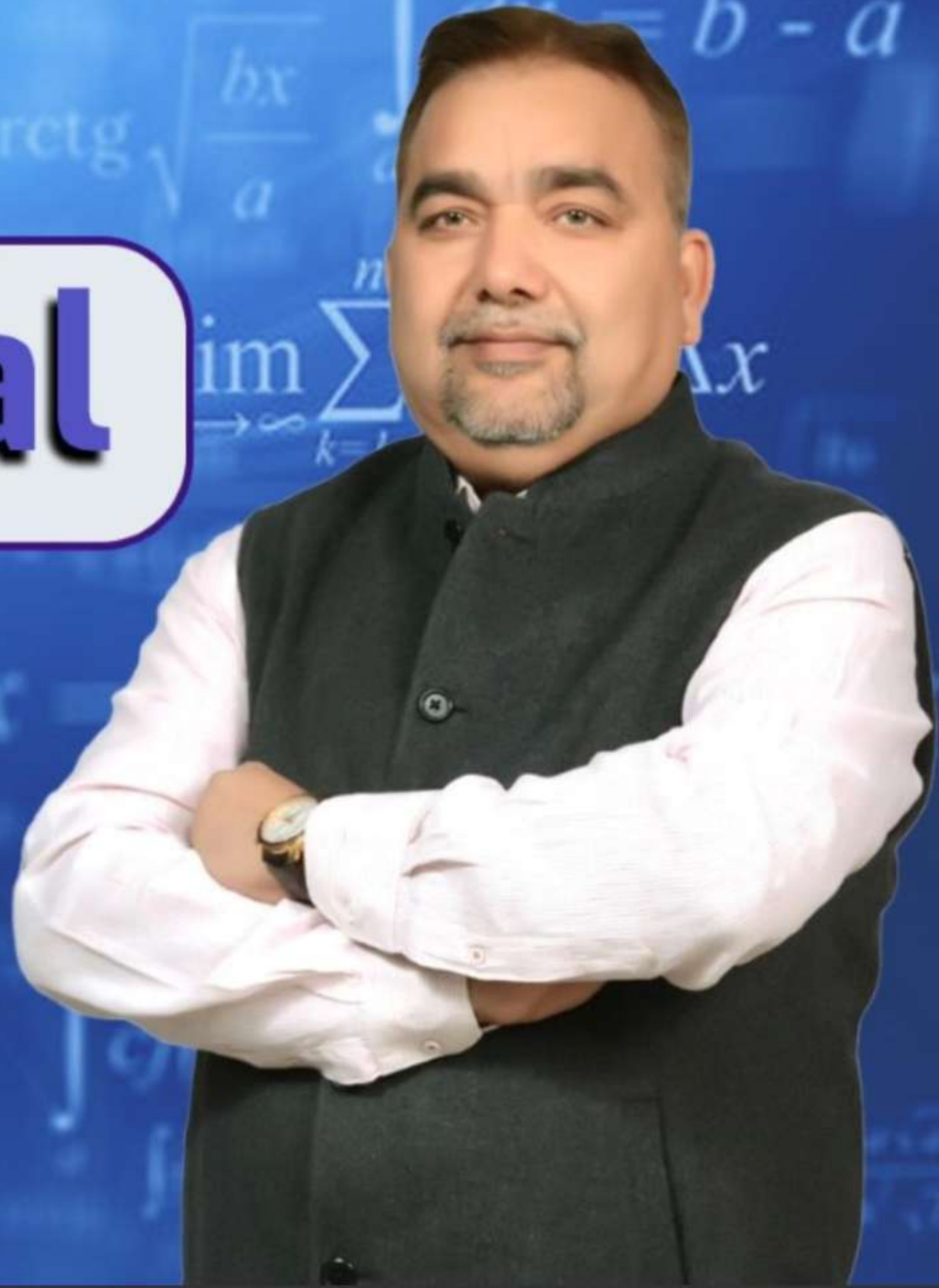

Th: $V(F)$ is a vector space, W is a subset of V; W is a subspace of V iff the internal and external compositions are satisfied in W i.e.,

(i) $\forall\, u, v \in W \Rightarrow u+v \in W$.

(ii) $\alpha \in F,\ u \in W \Rightarrow \alpha u \in W$.

Proof: Let W be a subspace of V.

$\therefore$ W is a vector space w.r.t the internal and external composition of V.

(i) Let $u, v \in W$

$\Rightarrow u+v \in W$ (closure prop.)

To prove: W is a vector space.

I (1) By condition (I),

$u+v \in W \quad \forall u, v \in W$

closure prop. holds.

(II) Let $u, v, w \in W \subseteq V$

$\therefore u+(v+w)$
$=(u+v)+w$
(Ass. holds in V)

(II) $\alpha \in F$, $u \in W$

$\Rightarrow \alpha u \in W$ will hold.

(W is a vector space)

Conversely let W be a subset of V which satisfies the two conditions.

(i.e) (I) & (II)

0 will act as identity of W.

(V) Take $a = -1$, & $v \in W$ in cond^n (II)

$-1 \cdot v \in W$

$\Rightarrow -v \in W$

Also $v + (-v) = (-v) + v = 0$

(inverse cond^n holds in V)

(III) Let $u, v \in W \subseteq V$

$\therefore u + v = v + u$

[commutativity holds in V]

(IV) Take $\alpha = 0$, $v \in W$ in condition (II),

$0 \cdot v = 0 \in W$

$\therefore 0 + v = v + 0 = v \quad \forall v \in W \subseteq V$

[identity cond^n holds in V]

$\therefore$ W is a vector space over F.

So W is a sub-space of V.

$\therefore (W, +)$ is an abelian group.

Let $\alpha, \beta \in F$, $u, v \in W \subseteq V$

$\therefore$

VI $\alpha(u+v) = \alpha u + \alpha v$.

VII $(\alpha+\beta)u = \alpha u + \beta u$

VIII $\alpha(\beta u) = (\alpha\beta)u$.

IX $1 \cdot u = u$

By ext. comp. conditions in V hold.

Let $a, b \in F$, $u, v, \in W$

$\Rightarrow au, bv \in W$

$\Rightarrow au + bv \in W$

(closure prop. holds for W)

Th: V(F) is a vector space, W is a subset of V; W is a subspace of V iff $a, b \in F$, $u, v \in W \Rightarrow au + bv \in W$.

Proof: Let W is a sub-space of V.

$\therefore$ Internal and external Composition will hold for W.

(ii) Let $u, v, w \in W \subseteq V$

$\therefore u+(v+w)$
$= (u+v)+w$

[By Ass. cond^n. in V]

(iii) Let $u, v \in W \subseteq V$

$\therefore u+v = v+u$

[By commutativity cond^n in V]

Conversely, suppose that

$a, b \in F$, $u, v \in W$
$\Longrightarrow au+bv \in W$ —— (1)

To prove: W is a subspace of V.

(i) Take $a=b=1$ in (1),

$1\cdot u + 1\cdot v = u+v \in W$
$\forall u, v \in W$

$\therefore$ closure prop. holds for W.

(V) $1 \in F \Rightarrow -1 \in F$

Take $a = -1$, $b = 0$

in our hypothesis

(I)

$$-1 \cdot u + 0 \cdot v \in W$$

$$\Rightarrow -u \in W$$

Also $u + (-u) = (-u) + u = 0$

$\therefore$ Inverse cond^n holds in W also. [By inverse cond^n in V]

(IV) Take $a = b = 0$ in cond^n (I),

$$0 \cdot u + 0 \cdot v = 0 + 0$$
$$= 0 \in W$$

$\therefore 0 + v = v + 0 = v \quad \forall v \in W \subseteq V$

[By identity cond^n in V]

$\therefore$ identity cond^n holds in W also.

$\therefore$ W is a vector space over the field F.

$\therefore$ W is a sub-space of V.

$\therefore (W, +)$ is an abelian group.

Let $\alpha, \beta \in F$, $u, v \in W \subseteq V$

(VI) $\alpha \cdot (u+v) = \alpha u + \alpha v$

(VII) $(\alpha+\beta) \cdot u = \alpha u + \beta u$

(VIII) $\alpha(\beta u) = (\alpha\beta) u$

(IX) $1 \cdot u = u$

due to Ext. comp. condn. hold in V.

$\therefore$ condition (I) holds.

By external composition in W,

$a \in F$, $u \in W$

$\Rightarrow au \in W$.

Conversely: Suppose that W, a subset of V satisfies conditions (I) & (II)

Th: $V(F)$ is a vector space, $W \subseteq V$;

W is a subspace of V iff

(I) $\forall u, v \in W \Rightarrow u - v \in W$

(II) $a \in F$, $u \in W \Rightarrow au \in W$.

proof: Suppose W is a subspace of V.

let $u, v \in W \Rightarrow u, -v \in W$

$\Rightarrow u + (-v) \in W$ [inverse ax^n. holds in W]

$\Rightarrow u - v \in W$ (closure prop holds in W)

(11) Take $u = 0$, $v \in W$

in condⁿ. (1)

$0 - v \in W$

$\Rightarrow -v \in W$

$\therefore v + (-v) = (-v) + v$

$= 0$

[By inverse condⁿ. in V]

$\therefore$ inverse condⁿ. holds in W also.

To prove: W is a subspace of V.

(1) Take $a = 0$, $u \in W$

in condition (11)

$0 \cdot w = 0 \in W$

Let $v \in W \subseteq V$

$\therefore 0 + v = v + 0 = v$

[By identity condⁿ. in V]

$\therefore$ identity condⁿ. holds in W also.

(IV) $u+v=v+u$

(V) $u+(v+w)=(u+v)+w$.

(VI)
(VII)
(VIII)
(IX)

(III) $u, v \in W$

$\Rightarrow u, -v \in W$

$\therefore$ By condⁿ. ①,

$u-(-v) \in W$

$\Rightarrow u+v \in W$

closure prop holds in W

$au \in W,\ v \in W$

$\Rightarrow \underline{au + v \in W}$

By closure prop. in W.

Conversely, $W \subseteq V$, satisfies the following condⁿ.

$a \in F,\ u, v \in W$

$\Rightarrow au + v \in W \Big\}$ —— (1)

Th: $V(F)$ is a vector space and $W \subseteq V$; W is a sub-space of V iff $a \in F,\ u, v \in W$

$\Rightarrow au + v \in W$.

Proof: Suppose W is a subspace of V.

Let $a \in F,\ u \in W$

$\Rightarrow au \in W$ [By ext. comp. condⁿ. in W]

(ii) $1 \in F, \Rightarrow -1 \in F$

Take $a = -1$, and $v = u$

in cond^n. (i),

$-u + u \in W$

$\Rightarrow 0 \in W$

but $v \in W \subseteq V$

$\therefore 0 + v = v + 0 = v$

$\therefore$ identity cond^n. holds in W also.

[By identity cond^n. in V]

<u>To prove</u>: W is a subspace of V.

(i) Take $a = 1$ in cond^n (i)

$1 \cdot u + v \in W, \quad \forall u, v \in W$

$\Rightarrow u + v \in W, \ \forall u, v \in W$

$\therefore$ closure prop. holds in <u>W</u>.

(IV) $u+(v+w)=(u+v)+w$

(V) $u+v=v+u$

(VI)
(VII)
(VIII)
(IX)

(III) Take $a=-1$, $u, v=0\in W$

in condⁿ. (1),

$-1\cdot u+0\in W$

$\Rightarrow -u\in W$

$\therefore u+(-u)=(-u)+u=0$

[By inverse condⁿ. in V]

$\therefore$ inverse condⁿ. holds in W also.

Note: Let $V(F)$ be a vector space.

$\{0\}$ — zero subspace

V

Subspace of V.

Improper sub-spaces

Prob: Let $W = \{(a_1, a_2, 0) : a_1, a_2 \in F\} \subseteq V_3(F)$. Then show that W is a subspace of $V_3(F)$.

Sol^n. Let $\alpha, \beta \in F$

Let $u = (a_1, a_2, 0)$
$v = (b_1, b_2, 0)$ $\in W$

$$\alpha u + \beta v$$
$$= \alpha(a_1, a_2, 0) + \beta(b_1, b_2, 0)$$
$$= (\alpha a_1, \alpha a_2, 0) + (\beta b_1, \beta b_2, 0)$$
$$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, 0) \in W$$

As $\left.\begin{matrix}\alpha a_1 + \beta b_1 \\ \alpha a_2 + \beta b_2\end{matrix}\right\} \in F$

Prob: prove that $W = \{(a, b, c) : a + b + 2c = 0,\ a, b, c \in \mathbb{R}\}$ is a subspace of the vector space $V_3(\mathbb{R})$ or $\mathbb{R}^3$.

Sol^n: Let $\alpha, \beta \in F$.

$u = (a_1, b_1, c_1) \in W$

$v = (a_2, b_2, c_2)$

$$\therefore \left.\begin{aligned} a_1 + b_1 + 2c_1 &= 0 \\ a_2 + b_2 + 2c_2 &= 0 \end{aligned}\right\} \text{---} \ (1)$$

$$\alpha u + \beta v = \alpha(a_1, b_1, c_1) + \beta(a_2, b_2, c_2)$$

$$= (\alpha a_1, \alpha b_1, \alpha c_1) + (\beta a_2, \beta b_2, \beta c_2)$$

$$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$$

Consider

$$\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2 + 2(\alpha c_1 + \beta c_2)$$

Th: The intersection of any two sub-spaces of a vector space $V(F)$ is also a sub-space of $V(F)$.

Proof: Let W_1 and W_2 be the subspaces of $V(F)$.

To prove:

$W_1 \cap W_2$ is a sub-space of $V(F)$.

$$= \alpha(\underbrace{a_1 + b_1 + 2c_1}) + \beta(\underbrace{a_2 + b_2 + 2c_2})$$

$$= \alpha \cdot 0 + \beta \cdot 0 \quad [\text{using } ①]$$

$$= 0 + 0$$

$$= 0$$

$\therefore \alpha u + \beta u \in W$

$\therefore W$ is a subspace of $V_3(\mathbb{R})$ or $\mathbb{R}^3$.

Note: The union of two sub-spaces of a vector space need not be a sub-space.

Consider the example

Let $W_1 = \{(x, 0) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$

$W_2 = \{(0, y) : y \in \mathbb{R}\} \subseteq \mathbb{R}^2$.

W_1 and W_2 are subspaces of $\mathbb{R}^2$ (Prove it)

Let $\alpha, \beta \in F$,

$u, v \in W_1 \cap W_2$

$\Rightarrow u, v \in W_1$, $u, v \in W_2$

As W_1 and W_2 are sub-spaces of V.

$\therefore \alpha u + \beta v \in W_1$, $\alpha u + \beta v \in W_2$

$\Rightarrow \alpha u + \beta v \in W_1 \cap W_2$

$\therefore W_1 \cap W_2$ is a subspace of V.

W_2 is a subspace of $\mathbb{R}^2$

(prove it)

$(1,0) \in W_1$, $(0,1) \in W_2$

i.e. $(1,0), (0,1) \in W_1 \cup W_2$

But $(1,0)+(0,1) = (1,1)$

$\notin W_1 \cup W_2$

Closure prop. does not hold for $W_1 \cup W_2$.

$\therefore W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$.

$W_1 = \{(x,0) : x \in \mathbb{R}\}$

Let $a, b \in \mathbb{R}$

$u = (x_1, 0)$
$v = (x_2, 0)$ $\in W_1$

$au + bv = a(x_1, 0) + b(x_2, 0)$

$= (ax_1 + bx_2, 0) \in W_1$

As $ax_1 + bx_2 \in \mathbb{R}$

$\therefore W_1$ is a subspace of $\mathbb{R}^2$.

Suppose $A \subseteq B$ or $B \subseteq A$

To prove: $A \cup B$ is a subspace of $V(F)$

If $A \subseteq B$ then

$$A \cup B = B$$

is a sub-sp.

If $B \subseteq A$ then

$$A \cup B = A$$

is a subspace

Th: The union of two sub-spaces is a subspace iff one is contained in the other.

proof: Let $V(F)$ be a vector space.

Let A and B are sub-spaces of V.

$\because B \nsubseteq A$.

$\therefore \exists\, v \in B$ s.t. $v \notin A$.

$\therefore u, v \in A \cup B$

$\Rightarrow u + v \in A \cup B$

(closure holds in $A \cup B$)

$\Rightarrow u + v \in A$ or $u + v \in B$

If $u + v \in A$, and $u \in A$

$\Rightarrow (u+v) - u \in A$ (A is a subsp.)

$\Rightarrow v \in A$

which is a contradiction

Conversely: Let A and B are subspaces of $V(F)$

Suppose: $A \cup B$ is a subspace

To prove: $A \subseteq B$ or $B \subseteq A$.

Let if possible,

$A \nsubseteq B$ and $B \nsubseteq A$

$\because A \nsubseteq B$

$\therefore \exists$ some $u \in A$ s.t. $u \notin B$.

If $u+v \in B$, & $v \in B$

$\Rightarrow (u+v) - v \in B$

[B is a sub-space]

$\Rightarrow u \in B$

which is a contradiction.

As $u \notin B$.

$\therefore$ our supposition is wrong.

So $A \subseteq B$ or $B \subseteq A$.