Practice questions (Linear Algebra)

① Let F be a subfield of complex numbers and T a function from $F^3 \to F^3$ defined by
$T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3, 2x_1 - x_2, -3x_1 + x_2 - x_3)$.
What are the conditions on a, b, c such that (a, b, c) be in the null space of T? Find the nullity of T.

IAS 2020
15M

Ans: $(a, b, c) = k(-1, -2, 1)$
Nullity $= 1$

② Let $A = \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix}$

(i) Find the rank of matrix A:

(ii) Find the dimension of the subspace

$$V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \right\}$$

IAS 2019
20M

Ans: Rank $= 2$,
$\dim V = 2$

③ Let $v_1 = (2,-1,3,2)$, $v_2 = (-1,1,1,-3)$ and $v_3 = (1,1,9,-5)$ be three vectors of the space $\mathbb{R}^4$. Does $(3,-1,0,-1) \in \text{span}\{v_1, v_2, v_3\}$? Justify your answer.

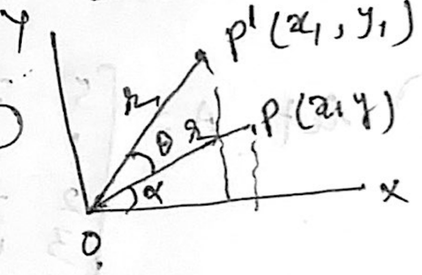
IAS 2023
10M

Ans: No.

④ Find a linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ which rotates each vector of $\mathbb{R}^2$ by an angle θ. Also prove that $\theta = \pi/2$, T has no eigen value in $\mathbb{R}$.

IAS 2022
15M

Hint:



$P(x,y) \in \mathbb{R}^2$

$x = r\cos\alpha \quad y = r\sin\alpha$ — ①

Rotating P through an angle θ, P becomes $p'(x_1, y_1)$

$x_1 = r\cos(\alpha+\theta) = x\cos\theta - y\sin\theta$

$y_1 = r\sin(\alpha+\theta) = x\sin\theta + y\cos\theta$

$T(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for $\theta = \pi/2$

$$T(x,y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = AX$$

$|A - \lambda I| = 0$

$\Rightarrow \lambda = \pm i \notin \mathbb{R}$.

(5.) Let the set $P = \{(x, y, z) : x - y - z = 0 \text{ \& } 2x - y + z = 0\}$

be the collection of vectors of a vector space $\mathbb{R}^3(\mathbb{R})$. Then

(i) Prove that P is a sub-space of $\mathbb{R}^3$.

(ii) Find a basis and dimension of P.

IAS 2022

20 M

Ans: $\dim P = 1$.

one of the basis of $P = \{-2, -3, 1\}$.

(6.) Suppose U and W are distinct four dimensional subspaces of a vector space V, where $\dim V = 6$. Find the possible dimensions of subspace $U \cap W$.

IAS 2017

10 M

Hint: $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$\Rightarrow \dim(U \cap W) = 8 - \dim(U+W)$

$U+W$ — a subspace of V.

possible dim. of $U+W = 4, 5, 6$

[$\because$ U and W are sub-spaces of $U+W$]

[$\dim U$ (or W) $\leq \dim(U+W)$]

$\therefore$ possible dim of $U \cap W = 4, 3, 2$

But $\dim U \cap W \neq 4$

otherwise, $U \cap W = U = W$ (But U and W are distinct).

(7.) If $W_1 = \{(x, y, z) : x + y - z = 0\}$

$W_2 = \{(x, y, z) : 3x + y - 2z = 0\}$

$W_3 = \{(x, y, z) : x - 7y + 3z = 0\}$

then find $\dim(W_1 \cap W_2 \cap W_3)$ and $\dim(W_1 + W_2)$.

IAS 2016

3 M

Ans: $\dim(W_1 \cap W_2 \cap W_3) = 1$

$\dim(W_1 + W_2) = 3$

(8.) Find the dimension of the subspace of $\mathbb{R}^4$, spanned by the set $\{(1, 0, 0, 0), (0, 1, 0, 0), (1, 2, 0, 1), (0, 0, 0, 1)\}$. Hence find its bases.

IAS 2015

12 M

Ans: 3

Basis $= \{(1, 0, 0, 0), (0, 1, 0, 0), (1, 2, 0, 1)\}$

(9.) Let V and W be the following subspaces of $\mathbb{R}^4$:

$V = \{(a, b, c, d) : b - 2c + d = 0\}$

& $W = \{(a, b, c, d) : a = d,\ b = 2c\}$

Find a bases and the dimension of (i) V (ii) W (iii) $V \cap W$.

IAS 2014

15 M

Ans: $\dim V = 3$, Basis $= \{(1,0,0,0), (0,1,0,-1), (0,0,1,2)\}$

$\dim W = 2$, Basis $= \{(1,0,0,1), (0,2,1,0)\}$

$\dim (V \cap W) = 1$, Basis $= \{(0,2,1,0)\}$.

(10) Find one vector in $\mathbb{R}^3$ which generates the intersection of V and W, where V is the xy-plane and W is the space generated by the vectors (1,2,3) and (1,-1,1)

IAS 2014
10 M

$\boxed{\text{Ans: } (-2,5,0)}$

Hints:

$V = \{(x,y,0) : x,y \in \mathbb{R}\}$

$W = \{a(1,2,3) + b(1,-1,1) : a,b \in \mathbb{R}\}$

$= \{(a+b, 2a-b, 3a+b) : a,b \in \mathbb{R}\}$

For $(x,y,z) \in V \cap W$

$3a+b = 0 = z \Rightarrow b = -3a$

$y = 2a-b \Rightarrow y = 5a$

$x = a+b \Rightarrow x = -2a$

$\therefore V \cap W = \{ \cancel{(-2a, 5a, 0) : a \in \mathbb{R}} \}$.

$\cancel{\{a(\ \{}$

$= \{(-2a, 5a, 0) : a \in \mathbb{R}\}$.

$= \{a(-2,5,0) : a \in \mathbb{R}\}$