KGS IAS

Maths Optional

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Prob: Show that $f_1(t) = 1$, $f_2(t) = t - 2$, $f_3(t) = (t-2)^2$ form a basis of P_2, the vector space of all polynomials with degree ≤ 2.

$\longrightarrow$ Let $S = \{f_1, f_2, f_3\}$

S.L.I

$$a f_1 + b f_2 + c f_3 = 0 \qquad a, b, c \in \mathbb{R}$$

$$a \cdot 1 + b(t-2) + c(t-2)^2 = 0$$

$$\Rightarrow a + bt - 2b + c[t^2 - 4t + 4] = 0$$

$$\Rightarrow ct^2 + (b - 4c)t + a - 2b + 4c = 0$$

$$= 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2$$

To prove $L(S) = P_2$

clearly, $L(S) \le P_2$

let $a_0 + a_1 t + a_2 t^2 \in P_2$

let $a_0 + a_1 t + a_2 t^2$

$= a \cdot f_1 + b \cdot f_2 + c \cdot f_3$

$= a \cdot 1 + b \cdot (t-2) + c \cdot (t-2)^2$

equating the coeff. of like powers of t,

$\underline{c = 0}$

$b - 4c = 0 \quad b = 0$

$a - 2b + 4c = 0$

put $c = 0,\ b = 0$

$a = 0$

$a-2b+4c=a_0$

$\Rightarrow a=a_0+2b-4c$

$=a_0+2(a_1+4a_2)$

$-4a_2$

$=a_0+2a_1+4a_2$

$\therefore a_0+a_1t+a_2t^2\in L(S)$

$\therefore P_2\subseteq L(S)$

so $P_2=L(S)$

$a_0+a_1t+a_2t^2=a+bt-2b$

$+c[t^2-4t+4]$

$=ct^2+(b-4c)t$

$+a-2b+4c$

Equating coeff of like powers of t,

$a_2=c$

$b-4c=a_1 \Rightarrow b-4a_2=a_1$

$b=a_1+4a_2$

Let, if, possible W is infinite dimensional

$\therefore$ W has an infinite basis S (say).

$$S \subseteq W \subseteq V$$

i.e. $S \subseteq V$

But $\dim V = n$.

$\therefore$ any subset of V can not have more than n L.I. vectors.

Dimension of a sub-space

Th: If W is a sub-space of a FDVS $V(F)$ then W is finite dimensional and $\dim W \le \dim V$.

Further $V = W \Leftrightarrow \dim V = \dim W$

Proof: Let $\dim V = n$.

W is a subspace of $V(F)$ and $V(F)$ is a FDVS.

be a basis of W.

$\therefore W \subseteq V$

And $S_1 \subseteq W$

$\therefore S_1 \subseteq V$

and S_1 is L.I.

$\therefore$ it can be extended to form the basis of V.

Let S_2 be the extension of S_1 where S_2 is the basis of V.

Here S is infinite and is L.I. which is a contradiction.

$\therefore$ our supposition is wrong.

$\therefore$ W is a finite dimensional subspace of V.

Let $\dim W = m$

To prove: $m \leq n$

Let $S_1 = \{w_1, w_2, \ldots w_m\}$

$\therefore \dim W \le \dim V$

and $\dim V \le \dim W$

Thus $\dim V = \dim W$

Conversely suppose

then $\dim V = \dim W$
$= n$ (say)

$\therefore \dim W = n$

and let S is a basis of W

$\therefore L(S) = W$

$\therefore$ No. of elements in S_1

$\le$ No. of elements in S_2

$\Rightarrow m \le n$

[$\because \dim V = n$
$\therefore S_2$ has n elements.]

i.e. $\dim W \le \dim n$

Further suppose $V = W$

$\therefore W$ is a subspace of V

and V is a subspace of W.

$S \subseteq W \subseteq V$

$\Rightarrow S \subseteq V$

and S has n linearly independent vectors.

$\therefore S$ is a basis of V

as $\underline{\dim V = n}$

$\therefore L(S) = V$

So $L(S) = W = V$

$\therefore \underline{V = W}$

$\because V$ is a FDVS.

$\therefore W_1, W_2, W_1+W_2, W_1 \cap W_2$ are finite dimensional.

Let $\dim(W_1 \cap W_2) = K$

Let $S = \{r_1, r_2, \ldots r_k\}$ be a basis of $W_1 \cap W_2$.

$\because W_1 \cap W_2 \subseteq W_1$

$\therefore S \subseteq W_1$

Th: If W_1 and W_2 are subspaces of a vector space $V(F)$ [FDVS], then

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Proof: $\because W_1$ and W_2 are sub-space of V.

$\therefore W_1+W_2,\ W_1 \cap W_2$ are also subspace of V.

the basis of W_2.

Let $S_2 = \{\gamma_1, \gamma_2, \ldots \gamma_k, \beta_1, \beta_2, \ldots \beta_t\}$

be a basis of W_2

$\therefore \dim W_2 = k+t$

Now $\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$= k+m+\cancel{k}+t-\cancel{k}$

$= k+m+t$

Now To prove $\dim(W_1+W_2) = k+m+t$

and S is a L.I subset of $W_1 \cap W_2$

$\therefore$ S can be extended to form the basis of W_1

Let $S_1 = \{\gamma_1, \gamma_2, \ldots \gamma_k, \alpha_1, \alpha_2, \ldots \alpha_m\}$ be the basis of W_1

$\therefore \operatorname{Dim} W_1 = k+m$.

Also S is a L.I subset of W_2

$\therefore$ It can be $[\because W_1 \cap W_2 \subseteq W_2]$

extended to form

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t$

$= -\big(c_1r_1 + \cdots + c_kr_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m\big)$ — (11)

$\downarrow$

L. comb. of elements of S_1

$\therefore$ R.H.S. of (11) $\in W_1$

$\therefore$ L.H.S of (11) $\in W_1$

For this, we will show that

$S_3 = \{r_1, r_2, \ldots r_k, \alpha_1, \alpha_2 \ldots \alpha_m, \beta_1, \beta_2, \ldots \beta_t\}$

is a basis of $S_1 + S_2$

<u>S_3 is L.I</u>

Let $c_1, c_2, \ldots c_k, a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_t \in F$ s.t.

$c_1r_1 + c_2r_2 + \cdots + c_kr_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \cdots b_t\beta_t = 0$ — (I)

$-(d_1 r_1 + \cdots + d_k r_k) + b_1 \beta_1 + \cdots + b_t \beta_t = 0$

$\therefore S_2 = \{r_1, r_2, \ldots r_k, \beta_1, \beta_2, \ldots \beta_t\}$

is L.I

$\therefore d_1 = d_2 = \cdots = d_k = b_1 = b_2 = \cdots = b_t = 0$

$\therefore$ (ii) $\Rightarrow$

$c_1 r_1 + \cdots + c_k r_k + a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_m \alpha_m = 0$

$b_1 \beta_1 + b_2 \beta_2 + \cdots + b_t \beta_t$

is a linear comb. of the element of S_2

$\therefore b_1 \beta_1 + b_2 \beta_2 + \cdots + b_t \beta_t \in W_2$ — (iii)

$\therefore b_1 \beta_1 + b_2 \beta_2 + \cdots + b_t \beta_t \in W_1 \cap W_2$

$\because S$ is a basis of $W_1 \cap W_2$

$\therefore \exists\, d_1, d_2, \ldots d_k \in F$ s.t

$b_1 \beta_1 + \cdots + b_t \beta_t = d_1 r_1 + d_2 r_2 + \cdots + d_k r_k$

<u>claim:</u> $L(S_3) = W_1 + W_2$

clearly, $L(S_3) \subseteq W_1 + W_2$

let $v \in W_1 + W_2$

$\Rightarrow v = \underline{w_i} + \underline{w_j}$ for some $w_i \in W_1$, $w_j \in W_2$

w_i: lin. comb. of element of S_1

w_j: Linear comb. of element of S_2

$\therefore v$ is a lin comb. of element of S_3

$\therefore S_1 = \{r_1, r_2, \ldots r_k, \alpha_1, \alpha_2, \ldots \alpha_m\}$ is L.I.

$\therefore c_1 = c_2 = \cdots = c_k = a_1 = a_2 = \cdots = a_m = 0$

$\therefore$ (1) $\Rightarrow$

$c_1 = c_2 = \cdots = c_k = a_1 = a_2 = \cdots a_m$

$= b_1 = b_2 = \cdots = b_t = 0$

so S_3 is L.I.

<u>Note</u>: If $W_1 \cap W_2 = \{0\}$

then

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \underbrace{\dim(W_1 \cap W_2)}_{0}$$

$$= \dim W_1 + \dim W_2.$$

$\therefore W_1 + W_2 \subseteq L(S_3)$

So $\underline{W_1 + W_2 = L(S_3)}$

Thus, S_3 is a basis of $W_1 + W_2$

$$\dim(W_1+W_2) = k+m+t$$

$$= \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$\therefore L(S_1) = A$

<u>S_1 is L.I:</u>

Let $a\cdot(1,0,0) + b(0,1,0) = (0,0,0)$

$\Rightarrow (a, b, 0) = (0,0,0)$

$\Rightarrow$ <u>$a = b = 0$</u>

$\therefore S_1$ is a basis of A

$\therefore \operatorname{Dim} A = 2$

<u>prob:</u> Let $A = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $B = \{(0, y, z) : y, z \in \mathbb{R}\}$ be two subspaces of $\mathbb{R}^3$. Find the dim. of $(A+B)$

$\rightarrow A = \{x(1,0,0) + y(0,1,0) : x, y \in \mathbb{R}\}$

Consider $S_1 = \{(1,0,0), (0,1,0)\}$.

$$\mathrm{Dim}(A+B) = \mathrm{Dim}\,A + \mathrm{Dim}\,B - \mathrm{Dim}(A\cap B)$$

$$= 2+2-1$$

$$= 3.$$

Similarly we can show that $S_2 = \{(0,1,0), (0,0,1)\}$ is a basis of B (prove it!)

Clearly $A \cap B = \{(0, y, 0) : y \in \mathbb{R}\}$

$S_3 = \{(0,1,0)\}$ is a basis of $A \cap B$.

$\therefore \mathrm{Dim}(A\cap B) = 1$ (prove it)

A and B are subspaces of $\mathbb{R}^4$ (prove it!)

HW

$S_1 = \{(1, 0, 0, 0), (0, 1, 0, 0)\}$

is a basis of A (prove it!)

$S_2 = \{(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

is a basis of B (prove it!)

Prob: Find the two subspaces A and B of $V = \mathbb{R}^4$ s.t. $\dim A = 2$, $\dim B = 3$ and $\dim(A \cap B) = 1$.

Sol^n

$A = \{(x, y, 0, 0) : x, y \in \mathbb{R}\}$

$B = \{(0, y, z, t) : y, z, t \in \mathbb{R}\}$.

Prob: Let $V = \mathbb{R}^4$

$$W = \left\{ (a, b, c, d) : \begin{array}{l} a = b + c, \\ c = b + d \end{array} \right\}.$$

Find a basis and the dimension of W.

$(a, b, c, d) \in W$, $\quad a = b + c$

$\quad c = b + d$

$$\therefore (a, b, c, d) = (b + c, b, b + d, d)$$

$$= (2b + d, b, b + d, d)$$

$$A \cap B = \{ (0, y, 0, 0) : y \in \mathbb{R} \}.$$

$$S_3 = \{ (0, 1, 0, 0) \}$$

is a basis of $A \cap B$ (prove it!)

$2a+b=0$

$a=0$ ✓

$a+b=0$

$b=0$ ✓

So S is a ~~basis~~ of W

$\therefore \operatorname{Dim} W = 2$

$= (2b, b, b, 0) + (d, 0, d, d)$

$= b(2, 1, 1, 0) + d(1, 0, 1, 1)$

let $S = \{(2, 1, 1, 0), (1, 0, 1, 1)\}$

$\therefore L(S) = W$

S is L-I

$a(2, 1, 1, 0) + b(1, 0, 1, 1)$ $= (0, 0, 0, 0)$

$\Rightarrow (2a+b, a, a+b, b) = (0, 0, 0, 0)$

Row - space of A

$$A \xrightarrow{R \cdot op.} \text{Echelon form}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} \qquad E = \begin{bmatrix} R_1' \\ R_2' \\ 0 \\ 0 \end{bmatrix}$$

$L(R_1', R_2')$

= Row-space of A

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

Lin. span $\{R_1, R_2, R_3\}$

= Row space.

Basis

Dimension !

$\underline{\text{Sol}^n}$:

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 2 & -1 & 4 & 0 \\ 4 & 1 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 4 & 1 & -1 & -3 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 3 & -9 & -3 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

Prob: Reduce that matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 2 & -1 & 4 & 0 \\ 4 & 1 & -1 & -3 \end{bmatrix}$$

to a row-reduced echelon form. Also find a basis for the row space and its dimension.

$\therefore$ S is a basis of Row. sp(A).

Dim(Row. sp.(A)) = 2

$$\sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

Row space of A

$= L\{(2, -1, 4, 0), (0, 1, -3, -1)\}$

Also $S = \{(2, -1, 4, 0), (0, 1, -3, -1)\}$ is L.I.