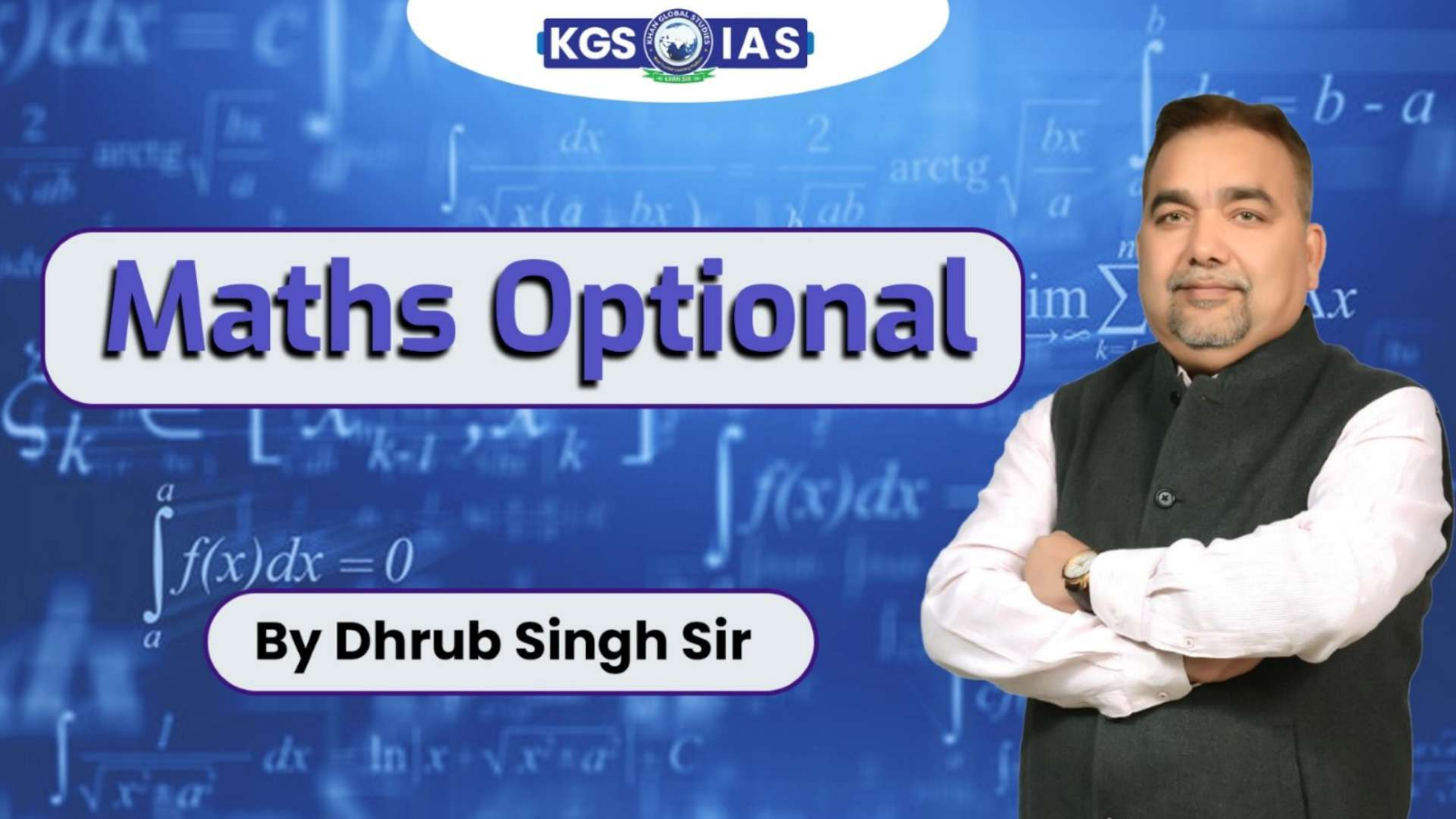
KGS IAS
Maths Optional
By Dhrub Singh Sir

(i) $\beta_1 = \{\overset{\alpha_1}{1}, \overset{\alpha_2}{x}, \overset{\alpha_3}{x^2}, \overset{\alpha_4}{x^3}\}$

(ii) $\beta_2 = \{\underset{\delta_1}{1}, \underset{\delta_2}{1+x}, \underset{\delta_3}{1+x^2}, \underset{\delta_4}{1+x^3}\}$

Solⁿ (i)

$$T(\alpha_1) = T(1) = 1$$
$$= 1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3$$
$$= 1\cdot\alpha_1 + 0\cdot\alpha_2 + 0\alpha_3 + 0\alpha_4$$

$$T(\alpha_2) = T(x) = x+1 = 1\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3$$
$$= 1\cdot\alpha_1 + 1\cdot\alpha_2 + 0\cdot\alpha_3 + 0\cdot\alpha_4$$

Prob: Let V be the vector space of polynomials of degree ≤ 3 over F. Let T be a linear operator defined on V as:

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$$
$$= a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3.$$

Compute the matrix of T relative to the bases

$$[T]_{\beta_1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

(11) $T(\delta_1) = T(1)$

$$= 1$$

$$= 1\cdot 1 + 0\cdot(1+x) + 0\cdot(1+x^2) + 0\cdot(1+x^3)$$

$$= 1\cdot\delta_1 + 0\cdot\delta_2 + 0\cdot\delta_3 + 0\cdot\delta_4$$

$$T(\alpha_3) = T(x^2) = (x+1)^2$$

$$= 1 + 2x + x^2$$

$$= 1\cdot 1 + 2\cdot x + 1\cdot x^2 + 0\cdot x^3$$

$$= 1\cdot\alpha_1 + 2\cdot\alpha_2 + 1\cdot\alpha_3 + 0\cdot\alpha_4$$

$$T(\alpha_4) = T(x^3) = (x+1)^3$$

$$= 1 + 3x + 3x^2 + x^3$$

$$= 1\cdot 1 + 3\cdot x + 3\cdot x^2 + 1\cdot x^3$$

$$= 1\cdot\alpha_1 + 3\cdot\alpha_2 + 3\cdot\alpha_3 + 1\cdot\alpha_4.$$

$$= -1 + 2(1+x) + 1\cdot(1+x^2)$$

$$= (-1)\cdot 1 + 2\cdot(1+x) + 1\cdot(1+x^2) + 0\cdot(1+x^3)$$

$$= (-1)\cdot \delta_1 + 2\cdot\delta_2 + 1\cdot\delta_3 + 0\cdot\delta_4$$

$$T(\delta_4) = T(1+x^3)$$

$$= 1 + (1+x)^3$$

$$= 1 + 1 + 3x + 3x^2 + x^3$$

$$= 2 - 3 + 3 + 3x - 3 + 3 + 3x^2 - 1 + 1 + x^3$$

$$T(\delta_2) = T(1+x)$$

$$= 1\cdot 1 + 1\cdot(1+x) + 0\cdot(1+x^2) + 0\cdot(1+x^3)$$

$$= 1\cdot\delta_1 + 1\cdot\delta_2 + 0\cdot\delta_3 + 0\cdot\delta_4$$

$$T(\delta_3) = T(1+x^2)$$

$$= 1 + 1\cdot(1+x)^2$$

$$= 1 + 1\left[1 + x^2 + 2x\right]$$

$$= 1 + 2x + 1\cdot(1+x^2)$$

$$= 1 - 2 + 2 + 2x + 1\cdot(1+x^2)$$

$$= -5 + 3(1+x) + 3(1+x^2) + 1\cdot(1+x^3)$$

$$= (-5)\cdot \delta_1 + 3\cdot \delta_2 + 3\cdot \delta_3 + 1\cdot \delta_4$$

$$[T]_{\beta_2} = \begin{bmatrix} 1 & 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$\{(1,1,1), (0,1,1), (0,0,1)\}$.

$\rightarrow$ Let $\beta = \{\overset{e_1}{(1,0,0)}, \overset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$T(e_1) = e_1 - e_2 = (1,-1,0)$

$T(e_2) = e_1 + 2e_2 + e_3 = (1,2,1)$

$T(e_3) = 2e_1 + e_2 + 3e_3 = (2,1,3)$

Prob: If the matrix of a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ is
$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$
the find the matrix of T relative to the basis

$$T(\alpha_1) = T(1,1,1) = (4,2,4)$$

$$= 4(1,1,1) - 2(0,1,1) + 2(0,0,1)$$

$$= 4\alpha_1 - 2\alpha_2 + 2\alpha_3$$

$$T(\alpha_2) = T(0,1,1)$$

$$= (3,3,4)$$

$$= 3(1,1,1) + 0\cdot(0,1,1) + 1\cdot(0,0,1)$$

$$= 3\alpha_1 + 0\cdot\alpha_2 + 1\cdot\alpha_3$$

$$T(x,y,z) = T(xe_1 + ye_2 + ze_3)$$

$$= x\,T(e_1) + y\cdot T(e_2) + z\cdot T(e_3)$$

$$= x(1,-1,0) + y(1,2,1) + z(2,1,3)$$

$$= (x+y+2z,\ -x+2y+z,\ y+3z)$$

Let $\beta_1 = \{\underset{\alpha_1}{(1,1,1)}, \underset{\alpha_2}{(0,1,1)}, \underset{\alpha_3}{(0,0,1)}\}$

$$T(\alpha_3) = T(0, 0, 1)$$

$$= (2, 1, 3)$$

$$= 2(1, 1, 1) - 1 \cdot (0, 1, 1)$$

$$+ 2(0, 0, 1)$$

$$= 2\alpha_1 - 1 \cdot \alpha_2 + 2 \cdot \alpha_3$$

$$[T]_{\beta_1} = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}_{3 \times 3}$$

Prob: Let T be a linear operator on $\mathbb{C}^2$ defined by $T(x_1, x_2) = (x_1, 0)$. Let $\beta = \{e_1 = (1,0), e_2 = (0,1)\}$, $\beta' = \{\alpha_1 = (1, i), \alpha_2 = (-i, 2)\}$ be ordered bases for $\mathbb{C}^2$. Determine

(i) $[T]_{\beta,\beta'}$ (ii) $[T]_\beta$ (iii) $[T]_{\beta',\beta}$
(iv) $[T]_{\beta'}$.

(i) $T(e_1) = T(1,0)$

$= (1,0)$

$(1,0) = a\,\alpha_1 + b \cdot \alpha_2$

$(1,0) = a(1,i) + b(-i,2)$

$= (a - bi,\ ai + 2)$

$a - bi = 1$ —— (1) $\times i$

$ai + 2b = 0$ —— (ii)

$ai + b = i$

$ai + 2b = 0$

$-b = i$

$$[T]_{\beta,\beta'} = \begin{bmatrix} 2 & 0 \\ -i & 0 \end{bmatrix}$$

(II) $$[T]_{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(III) $$[T]_{\beta',\beta} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

(IV) $$[T]_{\beta'} = \begin{bmatrix} 2 & -2i \\ -i & -1 \end{bmatrix}$$

$b = -i$

(1) $\Rightarrow$ $a - (-i)\cdot i = 1$

$\Rightarrow a - 1 = 1$

$\Rightarrow a = 2$

$\therefore T(e_1) = 2\alpha_1 - i\cdot\alpha_2$

$$\begin{aligned} T(e_2) &= T(0,1) \\ &= (0,0) \\ &= 0\cdot\alpha_1 + 0\cdot\alpha_2 \end{aligned}$$

$\rightarrow$ Let $\beta = \{e_1, e_2, e_3\}$ - St. basis of $\mathbb{R}^3$

$T(e_1) = e_1 - e_3 = (1, 0, -1)$

$T(e_2) = 2e_1 + e_2 + 3e_3$
$= (2, 1, 3)$

$T(e_3) = e_1 + e_2 + 4e_3$
$= (1, 1, 4)$

Prob: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator, the matrix A of which relative to the standard basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Find a basis for Range T and a basis for Null space of T.

$\in$ Range T

Range T = Span$\{(1,0,-1),(2,1,3),(1,1,4)\}$

Taking these vectors as the rows of a matrix.

$$\begin{bmatrix}1&0&-1\\2&1&3\\1&1&4\end{bmatrix} \sim \begin{bmatrix}1&0&-1\\0&1&5\\0&1&5\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix}1&0&-1\\0&1&5\\0&0&0\end{bmatrix} \quad R_3 \to R_3 - R_2$$

$T(x,y,z)$

$= T(xe_1 + ye_2 + ze_3)$

$= xT(e_1) + yT(e_2) + zT(e_3)$

$= x(1,0,-1) + y(2,1,3) + z(1,1,4)$

$= (x+2y+z,\ y+z,\ -x+3y+4z)$

$$x + 2y + z = 0$$
$$y + z = 0$$
$$-x + 3y + 4z = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

Basis of Range T

$$= \{ (1, 0, -1), (0, 1, 5) \}$$

Let $(x, y, z) \in N(T)$

$\therefore \quad T(x, y, z) = (0, 0, 0)$

$$\Rightarrow (x + 2y + z, \; y + z, \; -x + 3y + 4z)$$
$$= (0, 0, 0)$$

let $z = k$

$(II) \Rightarrow \; y = -k$

$(I) \Rightarrow \; x + 2(-k) + k = 0$

$x - k = 0$

$x = k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\to L.I.}$$

$N(T) = \{ k(1,-1,1) : k \in \mathbb{R} \}$

Basis $= \{ (1,-1,1) \}$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

rank of coeff. matrix $= 2$

$<$ No. of var.

$3 - 2 = 1$ — L.I. soln.

Now $x + 2y + z = 0$ — (I)

$y + z = 0$ — (II)

Find a matrix A which maps to a null matrix.

Soln: Let $A \in N(T)$

$\therefore T(A) = O$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a-c & b-d \\ -4a+4c & -4b+4d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Prob: Let $M_2(\mathbb{R})$ be the vector space of all 2×2 real matrices.

Let $B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$

Suppose $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is a L.T. defined by $T(A) = BA$. Find the rank and nullity of T.

rank of coeff matrix = 2

< No. of var.

$4 - 2 = 2$ — L.I. soln.

let $c = k_1$, $d = k_2$

$\therefore a = k_1 \quad b = k_2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 & 0 \\ k_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_2 \\ 0 & k_2 \end{bmatrix}$$

$a - c = 0$
$b - d = 0$

$$\left.\begin{aligned} -4a + 4c &= 0 \\ -4b + 4d &= 0 \end{aligned}\right\}$$

$\Rightarrow a - c = 0$
$b - d = 0$

$a - c = 0$
$b - d = 0$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By Rank-Nullity Th.

Rank T + Nullity $T = 4$ $\left\{ \operatorname{Dim} M_2(\mathbb{R}) = 4 \right\}$

Rank $T + 2 = 4$

Rank $T = 2$

$$= k_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} + k_2 \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}$$

$$N(T) = \left\{ \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix} : k_1, k_2 \in \mathbb{R} \right\}$$

Basis of $N(T) = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

$\operatorname{Dim} N(T) = 2$

Nullity $T = 2$

A must belong to $N(T)$

$$T(A) = 0$$

Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ ✓

$$\therefore T(A) = 0$$