Maths Optional

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Matrix Polynomial

$$F(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m$$

$A_0, A_1, A_2, \cdots A_m$ — matrices of order $n \times n$.

$A_m \neq 0$

Ex:

$$\begin{bmatrix} 1+2x & x^2 \\ 3x & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2x+0\cdot x^2 & 0+0\cdot x+1\cdot x^2 \\ 0+3\cdot x+0\cdot x^2 & 5+0\cdot x+0\cdot x^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^2$$

$$= \underline{A_0 + A_1 x + A_2 x^2}$$

CALEY HAMILTON Theorem

Statement: Every square matrix satisfies its characteristic eqn.

Proof: Let A be a square matrix of order n.
Let I be the identity matrix of order n.

The elements of ch. matrix $A-\lambda I$ will have at most 1 as the degree of λ.

$\therefore$ The elements of $adj(A-\lambda I)$ will be a polynomial in λ of degree at most $n-1$.

ch. poly of A is

$$|A-\lambda I|$$

$$= (-1)^n \left[\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n\right] \quad (1)$$

$\therefore$ Ch. eqn. of A is

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

To prove: $A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0$

$$= (-1)^n\left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n\right]\cdot I$$

using (I) and (II)

Comparing the coefficients of like powers of λ, we have

$$-B_0 = (-1)^n \cdot I$$

$$AB_0 - B_1 = (-1)^n a_1 I$$

$$AB_1 - B_2 = (-1)^n a_2 I$$

$$\therefore \text{Adj}(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1} \quad \text{— (II)}$$

where $B_0, B_1, B_2, \ldots B_{n-1}$ are matrices of order $n\times n$.

we have

$$(A-\lambda I)\cdot \text{Adj}(A-\lambda I) = |A-\lambda I|\cdot I$$

$$\Rightarrow (A-\lambda I)\left[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}\right]$$

$$O = (-1)^n \left[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I \right]$$

$$\Rightarrow A^n + a_1 A^{n-1} + \cdots + a_n I = O$$

$$AB_2 - B_3 = (-1)^n a_3 I$$

$$\vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots$$

$$AB_{n-1} = (-1)^n a_n I$$

Pre-Multiplying the above equations successively by $A^n, A^{n-1}, A^{n-2}, \ldots A, I$ and then adding, we have

$$A^{n-1}+a_1A^{n-2}+\cdots+a_n\cdot A^{-1}=0$$

$$\Rightarrow -a_nA^{-1}=A^{n-1}+a_1A^{n-2}+\cdots+a_{n-1}I$$

$$\Rightarrow A^{-1}=-\frac{1}{a_n}\left[A^{n-1}+a_1A^{n-2}+\cdots+a_{n-1}I\right]$$

Note: Let A is non-singular.

$$\therefore |A|\neq 0$$

Also $|A|=(-1)^n a_n$ [using (1) $\lambda=0$]

$$\Rightarrow (-1)^n a_n\neq 0$$

$$\Rightarrow a_n\neq 0$$

$$A^n+a_1A^{n-1}+a_2A^{n-2}+\cdots+a_nI=0$$

$$\Rightarrow A^{-1}\left[A^n+a_1A^{n-1}+a_2A^{n-2}+\cdots a_nI\right]=0$$

$= \lambda^2 - 5\lambda - 2$

$\therefore$ ch. eqn.

$\lambda^2 - 5\lambda - 2 = 0$

using CH Th.

we have

$A^2 - 5A - 2I = 0$

$\Rightarrow A^{-1}[A^2 - 5A - 2I] = 0$

$\Rightarrow A - 5I - 2A^{-1} = 0$

Prob: State C.H. Theorem and using its find the inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

Soln. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix}$

$= (1-\lambda)(4-\lambda) - 6$

$= 4 - \lambda - 4\lambda + \lambda^2 - 6$

$$2A^{-1} = A - 5I$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

Prob: Find the Ch. eqn. of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A and hence obtain its inverse.

Soln: $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Verification:

To verify $A^3 - 6A^2 + 9A - 4I = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3-6A^2+9A-4I=0$$

$$\Rightarrow A^{-1}\left(A^3-6A^2+9A-4I\right)=0$$

$$\Rightarrow A^2-6A+9I-4A^{-1}=0$$

$$\Rightarrow 4A^{-1}=A^2-6A+9I$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

Ch. eqn.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda) + 1 = 0$$

$$\Rightarrow 6 - 3\lambda - 2\lambda + \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 7 = 0$$

By C.H. Th,

$$A^2 - 5A + 7I = 0$$

Prob: State C.H. Th. use it to express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A when $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$.

Soln: Stat:

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}$$

$$= 55A - 126I$$

$$A^5 = A \cdot A^4$$
$$= A \cdot (55A - 126I)$$
$$= 55A^2 - 126A$$
$$= 55[5A - 7I] - 126A$$
$$= 275A - 126A - 385I$$
$$= 149A - 385I$$

$$A^2 = 5A - 7I$$

$$A^3 = A^2 \cdot A = 5A^2 - 7A$$
$$= 5(5A - 7I) - 7A$$
$$= 18A - 35I$$

$$A^4 = A \cdot A^3 = A(18A - 35I)$$
$$= 18A^2 - 35A$$
$$= 18(5A - 7I) - 35A$$

$$2A^5 - 3A^4 + A^2 - 4I$$

$$= 2\left[149A - 385I\right] - 3\left[55A - 126I\right]$$

$$+ 5A - 7I - 4I$$

$$= 138A - 403I$$

Prob: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then show that for every integer $n \geq 3$,

$$A^n = A^{n-2} + A^2 - I.$$

Hence find A^{50}.

Sol^n For $n = 3$

we will prove

$$A^3 = A + A^2 - I \quad \text{--- (1)}$$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Suppose that result is true for $n = k$.

$$\therefore A^{K} = A^{K-2} + A^{2} - I$$

$$\Longrightarrow A\,A^{K} = A\left(A^{K-2} + A^{2} - I\right)$$

$$\Longrightarrow A^{K+1} = A^{k-1} + A^{3} - A$$

$$= A^{k-1} + \cancel{A} + A^{2} - I - \cancel{A} \quad (\text{using } (i))$$

$$A^{K+1} = A^{(K+1)-2} + A^{2} - I$$

$\therefore$ Result holds for $n = k+1$.

So, by principle of mathematical induction. Result holds for all integer $n \geq 3$.

$= 3A^2 - 2I$

$\underline{n=8}$ $A^8 = A^6 + A^2 - I$

$= 3A^2 - 2I + A^2 - I$

$= 4A^2 - 3I$

proceeding in same manner.

$A^{50} = 25A^2 - 24I.$

(Put values in R.H.S.)

$\underline{A^{50}}$

$\therefore A^n = A^{n-2} + A^2 - I$

$\underline{n=3}$ $A^3 = A + A^2 - I$

$\underline{n=4}$ $A^4 = A^2 + A^2 - I$

$= 2A^2 - I$

$\underline{n=6}$ $A^6 = A^4 + A^2 - I$

$= 2A^2 - I + A^2 - I$