KGS IAS

Maths Optional

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(ii) $L(S) = F_2[x]$

clearly

$L(S) \subseteq F_2[x]$.

Let $a_0 + a_1 x + a_2 x^2 \in F_2[x]$

$a_0 + a_1 x + a_2 x^2$

$= a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2$

Lin. comb. of the element of S

$\Rightarrow a_0 + a_1 x + a_2 x^2 \in L(S)$

$\therefore L(S) = F_2[x]$

Ex: Let $F_2[x] = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in F\}$

$S = \{1, x, x^2\}$ is a basis of $F_2[x]$.

(1) S is L.I.

$a \cdot 1 + b \cdot x + c \cdot x^2 = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$a, b, c \in F$

$\Rightarrow \underline{a = b = c = 0}$

Finite dimensional vector space (FDVS):

The vector space $V(F)$ is said to be finite dimensional vector space if there exist a finite subset S of V s.t. $L(S) = V$.

Ex: $V_n(F) = F^n$.

$S = \{e_1, e_2, \ldots e_n\}$ is a basis of $V_n(F)$.

$e_1 = (1, 0, 0, \ldots 0)$

$e_2 = (0, 1, 0, \ldots 0)$

$\vdots$

$e_n = (0, 0, 0, \ldots 1)$

Ex: $\mathbb{R}^3$ is FDVS.

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$.

$L(S) = \mathbb{R}^3$

Note: If there does not exist a finite subset S of V s.t. $L(S) = V$ then V is called infinite dimensional vector space.

Ex: $V_2(F)$ is FDVS.

$S = \{e_1 = (1,0), e_2 = (0,1)\} \subseteq V_2(F)$

$L(S) = V_2$.

Infinite dimensional vector space: The vector space $V(F)$ is said to be infinite dimensional vector space if there exists an infinite subset S of V s.t. $L(S) = V$.

Ex: Show that the set $S = \{1, x, x^2, \cdots x^n\}$ is a basis for the vector space $F_n[x]$ of all polynomials of degree 'n' over the field F.

$\longrightarrow$ (Do it!)

$$a_1 \cdot x^{m_1} + a_2 \cdot x^{m_2} + \cdots + a_n \cdot x^{m_n} = 0$$
$$= 0 \cdot x^{m_1} + 0 \cdot x^{m_2} + \cdots + 0 \cdot x^{m_n}$$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$

$\therefore S_1$ is L.I

And so S is L.I

(II) $L(S) = F[x]$

Clearly, $L(S) \subseteq F[x]$.

Ex: Show that the set $S = \{1, x, x^2, x^3, \cdots\}$ is a basis of the vector space $F[x]$ of all polynomials over the field F.

Soln: (I) S is L.I.

Let $S_1 = \{x^{m_1}, x^{m_2}, \cdots x^{m_n}\}$ be a finite subset of S.

And so $L(S) = F[x]$

$\therefore S$ is a basis of $F[x]$.

Let $b_0 + b_1 \cdot x + b_2 \cdot x^2 + \cdots + b_n \cdot x^n \in F[x]$

$b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$

$= b_0 \cdot 1 + b_1 \cdot x + b_2 \cdot x^2 + \cdots$

$+ b_n \cdot x^n + 0 \cdot x^{n+1} + 0 \cdot x^{n+2} + \cdots$

Lin. comb. of elements of S

$\therefore b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \in L(S)$

$\therefore F[x] \subseteq L(S)$

proof: If S is L.I, then S is the basis of $V(F)$. And there is nothing to prove.

If S is L D then there exist a minimal subset T of S s.t. $L(T)=V$

i.e., $L(K) \neq V$

where K is a proper subset of T.

Th: Every FDVS $V(F)$ has a basis.

or

If $S=\{v_1, v_2, \dots, v_m\}$ spans $V(F)$ i.e., $L(S)=V$ then $\exists$ a subset of S which forms a basis of V.

$$(\alpha_i^{-1}\alpha_i)\, v_i = -(\alpha_i^{-1}\alpha_1 v_1 + \alpha_i^{-1}\alpha_2 v_2 + \cdots + \alpha_i^{-1}\alpha_{i-1} v_{i-1} + \alpha_i^{-1}\alpha_{i+1} v_{i+1} + \cdots + \alpha_i^{-1}\alpha_n v_n)$$

(1)

$v_i =$ " — (11)

Let $v \in V$ and $L(T) = V$

$\therefore\ v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_i v_i + \cdots + \beta_n v_n$

$\beta_i \in F$

Let $T = \{v_1, v_2, \cdots v_n\}$

Let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ — (1)

$\alpha_i \in F$

Let, if possible, $\exists$ some α_i

s.t. $\alpha_i \neq 0$

$\therefore\ \alpha_i^{-1}$ exist.

(1) $\Rightarrow$

$$\alpha_i v_i = -(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n)$$

$\therefore v \in L(K)$,

where $K = \{v_1, v_2, \ldots v_{i-1}, v_{i+1}, \ldots v_n\}$

which is a proper subset of T.

$L(K) = V$

which is a contradiction to our hypothesis that T is the minimal subset of S s.t. $L(T) = V$

$$v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{i-1} \cdot v_{i-1}$$
$$- \beta_i \left(\alpha_i^{-1}\alpha_1 v_1 + \alpha_i^{-1}\alpha_2 v_2 + \cdots + \alpha_i^{-1}\alpha_{i-1} v_{i-1} + \alpha_i^{-1}\alpha_{i+1} v_{i+1} + \cdots + \alpha_i^{-1}\alpha_n v_n \right)$$
$$+ \beta_{i+1} v_{i+1} + \cdots + \beta_n v_n$$

[using (11)]

$$\Rightarrow v = (\beta_1 - \beta_i \alpha_i^{-1} \alpha_1) v_1 + (\beta_2 - \beta_i \alpha_i^{-1} \alpha_2) v_2$$
$$+ \cdots + (\beta_{i-1} - \beta_i \alpha_i^{-1} \alpha_{i-1}) v_{i-1} + (\beta_{i+1} - \beta_i \alpha_i^{-1} \alpha_{i+1}) v_{i+1} + \cdots + (\beta_n - \beta_i \alpha_i^{-1} \alpha_n) v_n$$

$\therefore$ our supposition is wrong.

And so $\alpha_i = 0 \ \forall \ i = 1, \ldots n$

$(1) \Rightarrow$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$

$\therefore$ T is L.I.

So T is a basis of V.

Th: If $V(F)$ is a FDVS then any two bases of V have same number of element.

Proof: let $S = \{v_1, v_2, \ldots v_n\}$ and $T = \{w_1, w_2, \ldots \ldots w_m\}$ be the two bases.

To prove: $m = n$

suppose $n > m$

$S \subseteq V$

$v_i \in S \Rightarrow v_i \in V$

and T is a bases of V

$\therefore \exists \alpha_{ji} \in F$ s.t

$$v_i = \alpha_{1i} w_1 + \alpha_{2i} w_2 + \cdots + \alpha_{mi} w_m$$

$\forall i = 1, 2, \ldots n.$

$$(\alpha_{11}x_1+\alpha_{12}x_2+\cdots\alpha_{1n}x_n)w_1+$$
$$(\alpha_{21}x_1+\alpha_{22}x_2+\cdots+\alpha_{2n}x_n)w_2+$$
$$\cdots \quad \cdots \quad \cdots$$
$$(\alpha_{m1}x_1+\alpha_{m2}x_2+\cdots+\alpha_{mn}x_n)w_m$$
$$=0$$

$\because T=\{w_1, w_2, \cdots w_m\}$
is L.I.

Consider

$$x_1w_1+x_2w_2+\cdots+x_nw_n=0 \quad \text{— (1)}$$

$$x_i \in F$$

$$\Rightarrow x_1(\alpha_{11}w_1+\alpha_{21}w_2+\cdots+\alpha_{m1}w_m)+$$
$$x_2(\alpha_{12}w_1+\alpha_{22}w_2+\cdots+\alpha_{m2}w_m)+$$
$$\vdots \qquad \vdots$$
$$x_n(\alpha_{1n}w_1+\alpha_{2n}w_2+\cdots+\alpha_{mn}w_m)=0$$

Some x_i's will be non-zero.

From (1),

S is L.D.

i.e. a contradiction.

$\therefore$ our supposition is wrong.

$\therefore n \leq m$. — (11)

By taking $w_i \in T \subseteq V$ and S as a basis

$\therefore \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = 0$

$\alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n = 0$

$\vdots$

$\alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n = 0$

No. of eqns. $= m$

No. of variables $= n$

No. of variables > No. of eqns.

This system has non-zero soln. (As $n > m$)

is called the dimension of $V(F)$.

Notation: $\operatorname{Dim} V$.

or

$\operatorname{Dim}_F V$

Ex: (1) $\operatorname{Dim} \mathbb{R}^2 = 2$

$S = \{(1,0), (0,1)\}$

is a basis of $\mathbb{R}^2$.

Similarly, we can prove that

$m \leq n$ ——— (III)

By (II) & (III)

$m = n$.

Dimension of a vector space:

The number of element in any basis of a FDVS $V(F)$

(iv) $\operatorname{Dim}_{\mathbb{R}} \mathbb{C} = 2$.

$S = \{1, i\}$ is a basis of $\mathbb{C}(\mathbb{R})$.

(v) If F is any field then $\operatorname{Dim}_F F = 1$

$S = \{1\}$ is a basis.

(ii) $\operatorname{Dim} \mathbb{R}^3 = 3$

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$.

(iii) $\operatorname{Dim} \mathbb{R}^n = n$

$S = \{e_1, e_2, \ldots e_n\}$ is a basis of $\mathbb{R}^n$.

Prob: Determine whether or not the vectors $(1,-3,2)$, $(2,4,1)$, and $(1,1,1)$ forms a basis of $\mathbb{R}^3$.

$\longrightarrow$ $\underline{\operatorname{Dim} \mathbb{R}^3 = 3}$

$$a(1,-3,2)+b(2,4,1)+c(1,1,1) = (0,0,0)$$

$$\Rightarrow \begin{aligned} a+2b+c&=0 \\ -3a+4b+c&=0 \\ 2a+b+c&=0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & 4 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10 & 4 \\ 0 & -3 & -1 \end{bmatrix} \quad \begin{aligned} R_2 &\to R_2+3R_1 \\ R_3 &\to R_3-2R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10 & 4 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \quad R_3 \to R_3+\frac{3}{10}R_2$$

<u>prob:</u> Show that the set

$S = \{(1,0,0),\ (1,1,0),\ (4,5,0)\}$

is not a basis of $\mathbb{R}^3$.

HW

Rank of coeff. matrix

$= 3 =$ No. of variables.

$\therefore$ only one solⁿ.

i.e. trivial solⁿ.

$a = b = c = 0$

$\therefore \{(1,-3,2),\ (2,4,1),\ (1,1,1)\}$

is a L.I. subset of $\mathbb{R}^3$

$\therefore$ It is a basis of $\mathbb{R}^3$.

Consider the set:

$$S = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

S is LI:

$$a v_1 + b v_2 + c v_3 + d v_4 = 0$$

$a, b, c, d \in F$

Prob: Let V be the vector space of all 2×2 matrices over the field F. Prove that $\text{Dim } V = 4$ by finding a basis of V.

Sol^n:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in F \right\}$$

$\underline{L(S) = V}$

$L(S) \subseteq V$ (obvious)

let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a v_1 + b v_2 + c v_3 + d v_4$

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in L(S)$

$\therefore V \subseteq L(S)$

$\Rightarrow L(S) = V$

$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \underline{a = b = c = d = 0}$

$\therefore$ S is a basis of V.

So, $\underline{\text{Dim } V = 4}$.