KGS IAS

Maths Optional

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$\therefore \exists$ scalars $a_1, a_2, \ldots a_{i-1}, a_{i+1} \ldots - a_n \in F$

s.t.

$v_i = a_1 v_1 + a_2 v_2 + \ldots - a_{i-1} v_{i-1} + a_{i+1} v_{i+1} + \ldots + a_n v_n$

$\Rightarrow a_1 v_1 + a_2 v_2 + \ldots + (-1) v_i + \ldots a_n v_n = 0$

$\therefore S = \{v_1, v_2, - v_n\}$ is L.D. as $a_i = -1 \neq 0$

$\underline{\hspace{6em}}$ coeff of v_i

<u>Th:</u> Let $V(F)$ be a vector space. $S = \{v_1, v_2, \ldots - v_n\} \subseteq V$ if one of the vectors of S is the linear combination of all the remaining vectors then S is L.D.

<u>Proof</u>: $S = \{v_1, v_2, - - v_n\} \subseteq V$

Let v_i is the linear combination of the remaining vectors of S.

$\Rightarrow a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + (-1) w = 0$

$\therefore \{v_1, v_2, \ldots, v_n, w\}$ is L D as

coeff. of $w = -1 \neq 0$.

Th: If in a vector space $V(F)$, a vector w is a linear combination of the set of vectors $v_1, v_2, \ldots, v_n$ then the set of vectors $w, v_1, v_2, \ldots, v_n$ are L.D.

Proof: $\because$ w is a linear comb. of the vectors $v_1, v_2, \ldots, v_n$.

$\therefore \exists$ scalars $a_1, a_2, \ldots, a_n \in F$ s.t.

$$w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

$\alpha^{-1}(\alpha v + \alpha_1 v_1 + \cdots + \alpha_n v_n)$

$= \alpha^{-1} \cdot 0 = 0$

$\Rightarrow \underbrace{(\alpha^{-1}\alpha)}_{1} v + (\alpha^{-1}\alpha_1) v_1 + \cdots$

$+ (\alpha^{-1}\alpha_n) v_n = 0$

$\Rightarrow v = -\left[(\alpha^{-1}\alpha_1) v_1 + (\alpha^{-1}\alpha_2) v_2 + \cdots + (\alpha^{-1}\alpha_n) v_n\right]$

$\Rightarrow v \in L(S)$

which is a contradiction to the fact that $v \notin L(S)$.

Th: Let $V(F)$ be a vector space and a subset $S = \{v_1, v_2, \ldots, v_n\}$ of $V(F)$ be a linearly independent set. If $v \in V$ and $v \notin L(S)$ then prove that $S_1 = \{v, v_1, v_2, \ldots, v_n\}$ is L.I.

Proof: Consider

$$\alpha v + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \quad (1)$$

$\alpha, \alpha_i \in F$

If $\alpha \neq 0$, α^{-1} exists.

$\therefore$ our supposition is wrong.

$\therefore \alpha = 0$

$\therefore$ ① $\Rightarrow$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

[As $S = \{v_1, v_2, \cdots v_n\}$ is L.I.]

$\therefore \alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

$\therefore S_1 = \{v, v_1, v_2, \cdots v_n\}$ is L.I.

Suppose $v_1, v_2, \ldots v_n$ are L.D.

$\therefore \exists$ some scalars $a_1, a_2, \ldots a_n$ (not all zeros) s.t.

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0 \quad (1)$$

Suppose maximal value of i for which $a_i \neq 0$ is k.

i.e. $a_k \neq 0$ and $a_{k+1} = a_{k+2} = \cdots = a_n = 0$

Result: If $v_1, v_2, \ldots v_n$ are vectors in V, prove that either they are linearly independent or some v_k $(2 \leq k \leq n)$ is a linear combination of the preceding vectors $v_1, v_2, \ldots v_{k-1}$.

Proof: If $v_1, v_2, \ldots v_n$ are L.I then there is nothing to prove.

$\therefore$ ① $\Rightarrow$

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

$\because a_k \neq 0 \quad \therefore a_k^{-1}$ exists

$$a_k^{-1}(a_1 v_1 + a_2 v_2 + \cdots + a_k v_k) = a_k^{-1} \cdot 0 = 0$$

$$\Rightarrow (a_k^{-1} a_1) v_1 + (a_k^{-1} a_2) v_2 + \cdots + \underbrace{(a_k^{-1} a_k)}_{1} v_k = 0$$

$$\Rightarrow v_k = -\left[(a_k^{-1} a_1) v_1 + (a_k^{-1} a_2) v_2 + \cdots + (a_k^{-1} a_{k-1}) v_{k-1}\right].$$

$2 \leq k \leq n$

Prob: Let V be the vector space of the functions from $\mathbb{R} \to \mathbb{R}$. Show that $f, g, h \in V$ are L.I. where $f(t) = e^{2t}$, $g(t) = t^2$, $h(t) = t$.

Sol^n: Let $af(t) + bg(t) + ch(t) = 0$

$\Rightarrow ae^{2t} + bt^2 + ct = 0$ — (1) $a, b, c \in \mathbb{R}$

(1) is true for all $t \in \mathbb{R}$

In particular it is true for $t = 0, 1, 2$

$t = 0$ in (1)

$a = 0$

$t = 1$ in (1)

$a \cdot e^2 + b + c = 0$

$\Rightarrow b + c = 0$

$$ae^4 + 4b + 2c = 0$$

$a = 0$

$$4b + 2c = 0$$

$$\Rightarrow 2b + c = 0$$

Also

$$\begin{array}{r} b + c = 0 \\ - \quad - \quad \quad - \\ \hline b = 0 \end{array}$$

$$c = 0$$

$\therefore f, g, h$ are L.I.

Prob: If u, v, w are L.I. vectors of $V(F)$, where F is a field of Complex numbers then prove that $u+v$, $v+w$, $u+w$ are also L.I.

Sol^n: Let

$$a(u+v)+b(v+w)+c(u+w)=0 \qquad a,b,c \in F$$

$$\Rightarrow (a+c)u+(a+b)v+(b+c)w=0$$

$$\Rightarrow \begin{aligned} a+c&=0\\ a+b&=0\\ b+c&=0 \end{aligned} \qquad \left[\because u, v, w \text{ are L.I.}\right]$$

$$2(a+b+c)=0$$

$$\Rightarrow a+b+c=0$$

$$b+c=0$$

$$a=0$$

$||^{ly}$, $b=c=0$

$\therefore u+v, v+w, u+w$ are L.I.

Prob: Show that $\{(1+i, 2i), (1, 1+i)\} \subseteq \mathbb{C}^2(\mathbb{R})$ is L.I.

Sol^n: let $a(1+i, 2i) + b(1, 1+i) = (0, 0)$, $\quad a, b \in \mathbb{R}$

$$\Rightarrow (a + a\cdot i, 2ai) + (b, b + b\cdot i) = (0, 0)$$

$$\Rightarrow (a + b + a\cdot i, \; b + (b+2a)\cdot i) = (0, 0)$$

$$\Rightarrow a + b + a\cdot i = 0 + i\cdot 0, \qquad b + (b+2a)\cdot i = 0 + i\cdot 0$$

$\Rightarrow a + b = 0, \; a = 0$ | $b = 0, \; b + 2a = 0$

$\Rightarrow b = 0$ | $b = 0, \; a = 0$

$\therefore$ the given set is L.I.

Let $a_1 x^{m_1} + a_2 x^{m_2} + \cdots + a_n x^{m_n} = 0$

$a_i \in F$

i.e.

$$a_1 x^{m_1} + a_2 x^{m_2} + \cdots + a_n x^{m_n} = 0 \cdot x^{m_1} + 0 \cdot x^{m_2} + \cdots + 0 \cdot x^{m_n}$$

$\Rightarrow a_1 = 0, a_2 = 0, \ldots a_n = 0$

$\therefore S'$ is L.I.

and so S is L.I.

Prob: In the vector space $F[x]$ of all polynomials over the field F, the infinite set $S = \{1, x, x^2, \ldots\}$ is L.I.

Soln: Let $S' = \{x^{m_1}, x^{m_2}, \ldots x^{m_n}\} \subseteq S$.

To prove: S' is L.I.

$a+2b+c=2$

$+3a+4b+5c=+5$

$2a-b+7c=3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 4 & 5 & 5 \\ 2 & -1 & 7 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -2 & 2 & -1 \\ 0 & -5 & 5 & -1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

Prob: Is the vector $(2,-5,3)$ in the sub-space of $\mathbb{R}^3$ spanned by the vectors $(1,-3,2)$, $(2,-4,-1)$, $(1,-5,7)$?

Solⁿ:

Let

$$(2,-5,3) = a(1,-3,2) + b(2,-4,-1) + c(1,-5,7)$$

$a, b, c \in \mathbb{R}$

NO.

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & 3/2 \end{array}\right]$$

$R_3 \to R_3 - \frac{5}{2} R_2$

$r(\text{Coeff.}) = 2$

$r(\text{Aug. mat.}) = 3$

$\therefore r(\text{Coeff. M.}) \neq r(\text{Aug. mat.})$

$\therefore$ No value for a, b, c.

claim: w is a L.C. of u & v.

let $w = au + bv$, $a, b \in \mathbb{R}$

$$(3,-4,7) = a(1,2,1) + b(3,1,5)$$

$$a + 3b = 3 \quad \text{--- (ii)}$$
$$2a + b = -4 \quad \text{--- (iii)}$$
$$a + 5b = 7 \quad \text{--- (iv)}$$

$a = -3$, $b = 2$ (Solve it!)

prob: In the vector space $\mathbb{R}^3$, let $u = (1,2,1)$, $v = (3,1,5)$, $w = (3,-4,7)$. Show that the subspace spanned by $S = \{u, v\}$ and $T = \{u, v, w\}$ are the same.

Solⁿ: **To prove:** $L(S) = L(T)$

$S \subseteq T$

$\Rightarrow L(S) \subseteq L(T)$ --- (i)

$\therefore x \in L(S)$

$\therefore L(T) \subseteq L(S)$ — V

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$$L(S) = L(T)$$

$\therefore w = -3u + 2v$

Now we will prove

$$L(T) \subseteq L(S)$$

Let $x \in L(T)$

$$\Rightarrow x = au + bv + cw$$

$a, b, c \in \mathbb{R}$

$$= au + bv + c(-3u + 2v)$$

$$= (a - 3c)u + (b + 2c)v$$

$$\in L(S)$$

$\underline{S \text{ is L.I}}$

$$a e_1 + b e_2 = 0 \qquad a, b \in \mathbb{R}$$

$$\Rightarrow a(1,0) + b(0,1) = (0,0)$$

$$\Rightarrow (a, b) = (0, 0)$$

$$\Rightarrow a = 0,\ b = 0$$

$\underline{L(S) = \mathbb{R}^2}$

$L(S) \subseteq \mathbb{R}^2$ (obvious)

Let $(a, b) \in \mathbb{R}^2$

$$(a, b) = a(1,0) + b(0,1)$$
$$= a e_1 + b e_2$$

Basis: Let $V(F)$ be a vector space and $S = \{v_1, v_2, \ldots v_n\} \subseteq V$

If (1) S is L.I.

(11) $L(S) = V$

then S is a basis of V.

Ex (1) $S = \{e_1 = (1,0),\ e_2 = (0,1)\} \subseteq \mathbb{R}^2$

Ex: $S = \{e_1, e_2\}$ — basis of $\mathbb{R}^2$

Consider $T = \{(1,1), (1,0)\}$.

Claim: T is a basis of $\mathbb{R}^2$.

T is L.I:

$$a(1,1) + b(1,0) = (0,0)$$

$a, b \in \mathbb{R}$

$$\Rightarrow a+b=0, \quad a=0$$

$b=0$

$\therefore (a,b) \in L(S)$

$\therefore \mathbb{R}^2 \subseteq L(S)$

So, $L(S) = \mathbb{R}^2$

$\therefore S = \{e_1, e_2\}$ is a basis of $\mathbb{R}^2$.

standard bases.

Remark: A vector space may have more than one bases.

$\Rightarrow \quad (x,y) \in L(T)$

$\therefore \quad \mathbb{R}^2 \subseteq L(T)$

Thus: $L(T) = \mathbb{R}^2$.

$\therefore$ T is a basis of $\mathbb{R}^2$.

$L(T) = \mathbb{R}^2$

$L(T) \subseteq \mathbb{R}^2$ (clearly)

Let $(x,y) \in \mathbb{R}^2$

Let $(x,y) = a(1,1) + b(1,0)$

$a, b \in \mathbb{R}$.

$\Rightarrow a + b = x, \quad a = y$

$\Rightarrow b = x - a$

$b = x - y$

S is a basis of $\mathbb{R}^n$.

Ex: $S = \{e_1, e_2, \dots e_n\} \subseteq \mathbb{R}^n$

where $e_1 = (1, 0, 0, \dots 0)$

$e_2 = (0, 1, 0, \dots 0)$

$e_3 = (0, 0, 1, \dots 0)$

$\vdots$

$e_n = (0, 0, 0, \dots 1)$

S is L.I.

$L(S) = \mathbb{R}^n$.

Let $a+ib \in \mathbb{C}$

$$a+ib = a\cdot 1 + b\cdot i \quad a,b\in\mathbb{R}$$

$$\in L(S)$$

$\therefore \mathbb{C} \subseteq L(S)$

Thus $L(S) = \mathbb{C}$.

So S is a basis of $\mathbb{C}(\mathbb{R})$

Ex: $S=\{1, i\}$ is a basis of $\mathbb{C}(\mathbb{R})$

Sol^n: S is L.I.

$$a\cdot 1 + b\cdot i = 0 = 0 + i\cdot 0$$

$$\Rightarrow a=0,\ b=0.$$

$L(S) = \mathbb{C}$

$L(S) \subseteq \mathbb{C}$ (obvious)