KGS IAS

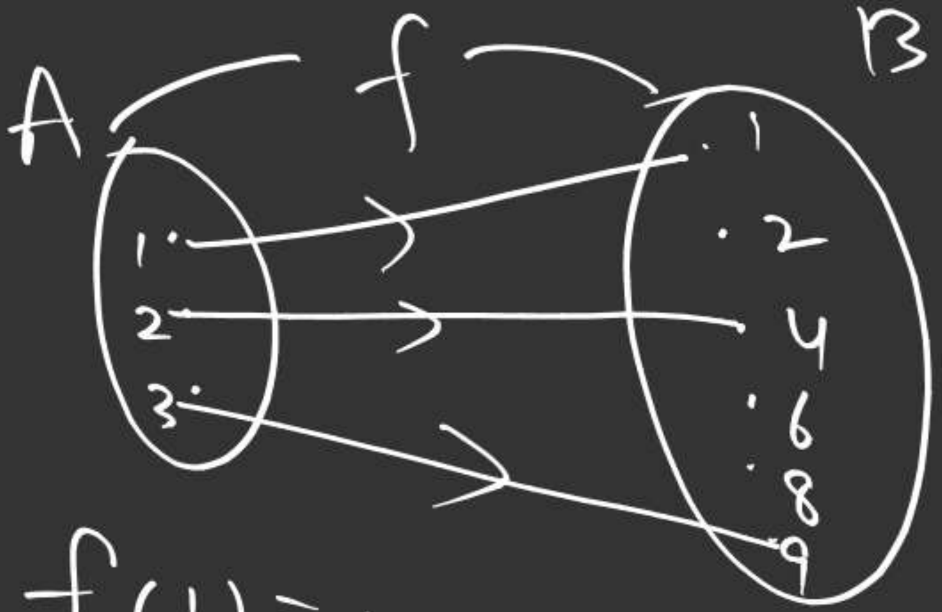
Maths Optional

By Dhrub Singh Sir

Vector space (सदिश समष्टि)

Ex $A = \{1, 2, 3\}$

$B = \{1, 2, 4, 6, 8, 9\}$



$f(1) = 1$

$f(2) = 4$ $\quad f(3) = 9$

$f(x) = x^2$

Function ?

A function f from A to B ($f : A \to B$) is a rule which associate **each** element of A to a **unique** element of B.

Binary operations

$\mathbb{N}$

$$3 + 4 = 7$$

$\in \mathbb{N}$ $\in \mathbb{N}$ $\in \mathbb{N}$

$$3 - 4 = -1 \notin \mathbb{N}$$

$\in \mathbb{N}$ $\in \mathbb{N}$

Set A = Domain of f

Set B = Co-Domain of f.

Range of $f = \{f(1), f(2), f(3)\}$

$= \{1, 4, 9\}$

Range $\subseteq$ Co-Domain

$* : A \times A \longrightarrow A$

$*(a, b) = a * b \in A$

$*$ — Binary operation.

$(A, *)$ — Algebraic structure.

$(A, \cdot)$

$\mathbb{N} \times \mathbb{N} = \{(a, b) : a, b \in \mathbb{N}\}$

$'+' : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$+(a, b) = a + b \in \mathbb{N}$

B.O.

B.O. $*, \circ$

Group (समूह)

Let $(G, *)$ be an algebraic structure. It is called a group if the following conditions are satisfied:

(I) $a * b \in G \;\; \forall\, a, b \in G$

(closure)

(II) $a * (b * c) = (a * b) * c \;\; \forall\, a, b, c \in G.$

(Associative)

(III) $\exists$ an element e (called identity) $\in G$ s.t.

$$a * e = e * a = a \quad \forall\, a \in G.$$

(IV) For each $a \in G$ $\exists$ an element $b \in G$ s.t.

$$a * b = b * a = e$$

b — inverse of a. $\rightarrow$ (a^{-1})

Ex: $(\mathbb{R}, +)$ — Abelian group

$\to$ Simple addition.

(I) $a+b \in \mathbb{R}, \ \forall a,b \in \mathbb{R}$

(II) $a+(b+c) = (a+b)+c$
$\forall a,b,c \in \mathbb{R}$

(III) $0 \in \mathbb{R}$
$a+0 = 0+a = a \ \forall a \in \mathbb{R}$

(IV) For each $a \in \mathbb{R}$, $\exists -a \in \mathbb{R}$
s.t $a+(-a) = -a+a = 0$

(V) $a+b = b+a \ \forall a,b \in \mathbb{R}$

Note (I) $a * b \to a \cdot b$ $\quad$ ab

(II) G is a group.

$ab = ba \quad \forall a,b \in G.$

G — abelian group.
(Commutative group).

Ex: $(\mathbb{Z}^*, \cdot)$ $\mathbb{Z}^* = \mathbb{Z} \sim \{0\}$

$2 \in \mathbb{Z}^*$

inverse of 2 doesn't exist.

$\frac{1}{2} \times 2 = 1$

$\frac{1}{2} \notin \mathbb{Z}^*$.

Not a group.

Ex: $(\mathbb{Z}, +)$ — Abelian group.

Ex: $(\mathbb{R}^*, \cdot)$ $\mathbb{R}^* = \mathbb{R} \sim \{0\}$

simple multiplication.

1 — identity

$\frac{1}{a}$ — inverse of a.

Abelian group.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G.$

$\rightarrow -A \in G.$

$A + (-A) = (-A) + A$

$\underbrace{-A}_{\text{inverse of } A} = O$

Abelian group

Ex: $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$

'+' — addition of matrices.

$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow$ Identity.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + O = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Ex: $(\mathbb{R}, +, \cdot)$ — Field.

$+$ → simple addition

$\cdot$ → " multip.

(I) $(\mathbb{R}, +)$ Abelian group.

(II) $(\mathbb{R}^*, \cdot)$ "

(III) $a(b+c) = ab + ac$ $\forall a, b, c \in \mathbb{R}$

Field (क्षेत्र)

$(F, +, \cdot)$ is a field if

(I) $(F, +)$ abelian group.

(II) $(F^*, \cdot)$ abelian group.

$F^* = F \sim \{0\}$

(III) $a \cdot (b+c) = ab + ac$ $\forall a, b, c \in F$

Sub-field:

Let $(F,+,\cdot)$ be a field.
A set $K \subseteq F$ is called a sub-field of F if K is itself a field under the operations of F.

Ex: $(\mathbb{Q},+,\cdot)$ — field

$\mathbb{Q}$ → set of rational No.

Ex: $(\mathbb{C},+,\cdot)$ — field

$\mathbb{C}$ → set of complex no.

Ex: $(\mathbb{Z},+,\cdot)$ — NOT a field.

Internal composition

Let A be a set.

If $a * b \in A \quad \forall a, b \in A$

then '$*$' — internal composition on A.

External composition

Let V and F be two set.

Ex: $(\mathbb{Q}, +, \cdot)$ is a sub-field of $(\mathbb{R}, +, \cdot)$

Ex: $(\mathbb{R}, +, \cdot)$ is a sub-field of $(\mathbb{C}, +, \cdot)$

$$2 \cdot \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} \in G.$$

'$\cdot$' — external composition in G over $\mathbb{R}$ $(=F)$.

If $a * v \in V$

$\forall a \in F, \forall v \in V$

then '$*$' — external composition in V over F.

Ex: $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$

$F = \mathbb{R}$

$2 \in F$, $\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \in G.$

Vector space (Linear space)

Let V be a non-empty set and F be a field.

Introduce two compositions as follows:

(A) Denote a binary operation in V as '$+$' so that

$$u+v \in V \quad \forall u, v \in V.$$

(B) For each $\alpha \in F$, & $v \in V$,

$$\alpha \cdot v \in V$$

'$\cdot$' — external composition in V over F. (Scalar multiplication).

V is called a vector space over F if the following conditions are satisfied:

I. $u + v \in V \quad \forall\, u, v \in V$

II. $u + (v + w) = (u + v) + w \quad \forall\, u, v, w \in V$

III. $u + v = v + u \quad \forall\, u, v \in V.$

IV. $\exists$ an element $\underline{0}$ in V s.t. $u + 0 = 0 + u = u \quad \forall\, u \in V$

V. For each $v \in V$, $\exists\, -v \in V$ s.t.

$$v + (-v) = (-v) + v = 0$$

VI. $\alpha(u + v) = \alpha u + \alpha v \quad \forall\, \alpha \in F,\ \forall\, u, v \in V$

VII. $(\alpha + \beta) v = \alpha v + \beta v \quad \forall\, \alpha, \beta \in F,\ \forall\, v \in V$

(II) Due to first five properties,

$(V, +)$ — Abelian group.

(III) $F = \mathbb{R}$

V(ℝ): Real vector space.

$F = \mathbb{C}$

V(ℂ): Complex vector space

(VIII) $\alpha(\beta v) = (\alpha\beta)\cdot v$

$\forall\, \alpha, \beta \in F$

$\forall\, v \in V$

(IX) $1\cdot v = v$

$1 \in F$

$\forall\, v \in V$

Note (1) V(F)

Elements of V — vectors.

Elements of F — scalars.

NO.

$(\mathbb{Z}, +)$ is not a vector space over $\mathbb{Q}$.

Prob: $V = (\mathbb{Z}, +)$ $F = \mathbb{Q}$

Is $V(F)$ a vector space?

Internal comp: simple addition.

External comp: Simple mult.

$\frac{1}{2} \in \mathbb{Q}$, $3 \in \mathbb{Z}$.

$\frac{1}{2} \times 3 = \frac{3}{2} \notin \mathbb{Z}$.

Let $\alpha, \beta \in \mathbb{Q}$, $u, v \in \mathbb{R}$

(II) $\alpha(u+v)$
$= \alpha u + \alpha v$

Distributive Law holds in $\mathbb{R}$.

(III) $(\alpha+\beta)u$
$= \alpha u + \beta u$

(IV) $\alpha(\beta u) = (\alpha\beta)u$

Associativity holds in $\mathbb{R}$.

Prob: $V = (\mathbb{R}, +)$ $F = \mathbb{Q}$.

I $u, v \in \mathbb{R}$, $u+v \in \mathbb{R}$

II $\alpha \in \mathbb{Q}$, $v \in \mathbb{R}$

$\alpha v \in \mathbb{R}$.

(1) $(\mathbb{R}, +)$ — Abelian group.

Ex: $\mathbb{R}(\mathbb{Q})$, $\mathbb{C}(\mathbb{R})$ } vector space.

Result: A field K can be regarded as a vector space over any subfield F.

proof: $\because (K, +, \cdot)$ is a field

$\therefore (K, +)$ — Abelian group.

(V) $1 \cdot u = u$ $\quad 1 \in \mathbb{Q}$

$\forall\, u \in \mathbb{R}$

$(\mathbb{R}, +)$ is a vector space over $\mathbb{Q}$.

Note: If $F \subseteq V$ then V is a vector space over F.

(IV) $1 \in F$,

$1 \cdot \alpha = \alpha \quad \forall \alpha \in K$.

$\therefore$ K is a vector space over F.

Let $a, b \in F (\subseteq K)$, $\alpha, \beta \in K$

$\therefore a, b, \alpha, \beta \in K$

(I) $a(\alpha + \beta) = a\alpha + a\beta$.

Distributive prop. holds in K.

(II) $(a+b)\alpha = a\alpha + b\alpha$

(III) $a(b\alpha) = (ab)\alpha$.

Associativity holds in K.

Scalar multiplication of matrices.

I $(V, +)$ — abelian group.

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ — identity}$$

A —— $-A$.

$$A + (-A) = O$$

Examples

① V = set of all 2×3 matrices with their elements as real numbers.

$F = \mathbb{R}$

Internal composition

Sum of matrices.

External comp:

Let $\alpha, \beta \in \mathbb{R}$, $A, B \in V$.

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix}$$

1. $\alpha(A+B)$

$$= \alpha \begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_4+b_4 & a_5+b_5 & a_6+b_6 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_1+\alpha b_1 & \alpha a_2+\alpha b_2 & \alpha a_3+\alpha b_3 \\ \alpha a_4+\alpha b_4 & \alpha a_5+\alpha b_5 & \alpha a_6+\alpha b_6 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_1 & \alpha a_2 & \alpha a_3 \\ \alpha a_4 & \alpha a_5 & \alpha a_6 \end{bmatrix} + \begin{bmatrix} \alpha b_1 & \alpha b_2 & \alpha b_3 \\ \alpha b_4 & \alpha b_5 & \alpha b_6 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} + \alpha \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix}$$

$$= \alpha A + \alpha B.$$

(II) $(\alpha+\beta)A = \alpha A + \beta A$

(prove it)

(III) $\alpha(\beta A) = (\alpha\beta)A$ 〃

(IV) $1 \cdot A = A$, $1 \in \mathbb{R}$

$\therefore$ V is a vector space over $\mathbb{R}$.