KGS IAS

Maths Optional

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$\therefore$ T is singular.

$\therefore$ T is not invertible.

Prob: Find if the L.T. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x+y, x+y)$ is invertible?

$\to$ consider $(1, -1)$

$$T(1, -1) = (1-1, 1-1)$$
$$= (0, 0)$$

$$\therefore (1, -1) \in N(T)$$

$$\therefore N(T) \neq \{0\}$$

is a standard basis of $\mathbb{C}^3$.

But $\{T(e_1), T(e_2), T(e_3)\}$ is not a basis of $\mathbb{C}^3$.

Let $aT(e_1) + bT(e_2) + cT(e_3) = (0,0,0)$ $\quad a,b,c \in \mathbb{C}$

$\Rightarrow a(1,0,i) + b(0,1,1) + c(i,1,0) = (0,0,0)$

<u>Prob</u>: Let T be a linear operator on $\mathbb{C}^3$ defined by

$T(1,0,0) = (1,0,i)$

$T(0,1,0) = (0,1,1)$

$T(0,0,1) = (i,1,0)$.

Is T invertible? Justify your answer.

$\rightarrow$ Here $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$

$$\sim \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

rank(Coeff matrix) = 2

$3-2 = 1$ L.I. soln.

$$a + ci = 0 \qquad b + c = 0$$

$c = i$ (say)
$b = -i$
$a = 1$

$\Rightarrow a + c \cdot i = 0$ — (I)

$b + c = 0$ — (II)

$\underline{ai + b = 0}$ — (III) $\times i$

$$\begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - iR_1$

Prob: let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a L.T. and let $U:\mathbb{R}^2 \to \mathbb{R}^3$ be a L.T. prove that the transformation UT is not invertible.

$\longrightarrow$ $UT:\mathbb{R}^3 \to \mathbb{R}^3$

let, if possible, UT is invertible.

$\therefore N(UT) = \{0\}$.

$\therefore \{T(e_1), T(e_2), T(e_3)\}$ is L.D. and is not a basis of $\mathbb{C}^3$.

$\therefore$ T is not non-singular

$\therefore$ T is not invertible.

$\therefore N(T)=\{0\}$.

$\therefore$ Nullity $T = 0$

By Rank-Nullity th.,

Rank T + Nullity $T = 3$

$\Rightarrow$ Rank $T + 0 = 3$

$\Rightarrow$ Rank $T = 3$

which is a contradiction.

<u>Claim:</u> $N(T)=\{0\}$.

Let $v \in N(T)$

$\Rightarrow T(v)=0$

$\Rightarrow U(T(v)) = U(0)$

$\Rightarrow UT(v) = 0$ [$\because U$ is $\ell.T$]

$\Rightarrow v \in N(UT)$

$\because N(UT)=\{0\}$.

$\therefore \underline{v=0}$

$\therefore$ Range(T) is a subspace of $\mathbb{R}^2$.

$$\mathrm{Dim}\big(R(T)\big) \leq 2$$

$\therefore$ Rank $T \leq 2$

$\therefore$ our supposition is wrong.

So, UT is ~~not~~ invertible.

Matrix of a linear transformation:

Let $V(F)$ & $W(F)$ be two finite dimensional vector spaces.

Let $\dim V = n$ & $\dim W = m$

Let $\beta = \{v_1, v_2, \ldots v_n\}$ be the ordered basis of V and

Let $\beta' = \{w_1, w_2, \ldots w_m\}$ be the ordered basis of W.

Let $T : V \to W$ be a L.T.

$v_j \in V \quad \therefore T(v_j) \in W$

β' is a basis of W.

$$T(v_j) = \alpha_{1j} w_1 + \alpha_{2j} w_2 + \cdots + \alpha_{mj} w_m$$

$\forall\ j = 1, 2, \ldots n$

$\alpha_{ij} \in F$

$[T]_{\beta,\beta'}$ — Matrix of L.T.

T w.r.t. bases β & β'.
(relative to)

<u>Note</u>: If $W = V$ and if β and β' are two order bases of V, then for any linear operator T on V, we obtain two matrices $[T]_{\beta,\beta'}$ or $[T]_{\beta',\beta}$

$$T(v_1) = \alpha_{11}w_1 + \alpha_{21}w_2 + \cdots + \alpha_{m1}w_m.$$
$$T(v_2) = \alpha_{12}w_1 + \alpha_{22}w_2 + \cdots + \alpha_{m2}w_m.$$
$$T(v_3) = \alpha_{13}w_1 + \alpha_{23}w_2 + \cdots + \alpha_{m3}w_m.$$
$$\vdots$$
$$T(v_n) = \alpha_{1n}w_1 + \alpha_{2n}w_2 + \cdots + \alpha_{mn}w_m$$

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}_{m\times n} = [T]_{\beta,\beta'}$$

& $\beta' = \{ \underset{\delta_1}{(0,1)}, \underset{\delta_2}{(1,0)} \}$ be the ordered bases of $\mathbb{R}^3$ & $\mathbb{R}^2$ resp. then find $[T]_{\beta,\beta'}$. Also find rank T and Nullity T.

$\longrightarrow$

$T(\alpha_1) = T(1,0,-1) = (1,-3) = -3(0,1) + 1\cdot(1,0)$
$= -3\cdot\delta_1 + 1\cdot\delta_2$

$T(\alpha_2) = T(1,1,1) = (2,1) = 1\cdot(0,1) + 2\cdot(1,0)$
$= 1\cdot\delta_1 + 2\cdot\delta_2$

$T(\alpha_3) = T(1,0,0) = (1,-1) = -1\cdot\delta_1 + 1\cdot\delta_2$

If $\beta = \beta'$,

$[T]_{\beta,\beta}$ as $[T]_\beta$.

Prob: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a L.T. defined by

$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$

If $\beta = \{ \underset{\alpha_1}{(1,0,-1)}, \underset{\alpha_2}{(1,1,1)}, \underset{\alpha_3}{(1,0,0)} \}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

Rank $= 2 <$ No. of var.

$3-2 = 1$ — L.I. soln.

$x+y=0$

$-y-2z=0 \Rightarrow y+2z=0$

$$[T]_{\beta,\beta'} = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}_{2\times 3}$$

Let $v=(x,y,z)\in N(T)$

$\therefore T(x,y,z)=(0,0)$

$\Rightarrow (x+y, 2z-x)=(0,0)$

$\Rightarrow x+y=0$

$x-2z=0$

$N(T) = \{K(2,-2,1) : K \text{ is arb.}\}$

$\text{Basis} = \{(2,-2,1)\}$.

$\text{Nullity } T = 1$

By Rank-Nullity th,

$\text{Rank } T + \text{Nullity } T = \text{Dim } \mathbb{R}^3$

$\text{Rank } T + 1 = 3$

$\text{Rank } T = 2$

Let $z = K$ (arb.)

$y = -2K$

$x = -y = 2K$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2K \\ -2K \\ K \end{bmatrix} = K \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

L.I

$$T(e_1) = T(1,0) = (1, 2, 0)$$
$$= 1\cdot\alpha_1 + 2\cdot\alpha_2 + 0\cdot\alpha_3$$

$$T(e_2) = T(0,1) = (1, -1, 7)$$
$$= 1\cdot\alpha_1 + (-1)\cdot\alpha_2 + 7\cdot\alpha_3$$

$$[T]_{\beta_1, \beta_2} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}_{3\times 2}$$

<u>Prob</u>: $T: V_2 \to V_3$ be defined by $T(x, y) = (x+y, 2x-y, 7y)$. Find $[T]_{\beta_1, \beta_2}$ where β_1 and β_2 are standard bases of V_2 & V_3 resp.

$\rightarrow$ $\beta_1 = \{e_1 = (1,0),\ e_2 = (0,1)\}$

$\beta_2 = \{\alpha_1 = (1,0,0),\ \alpha_2 = (0,1,0),\ \alpha_3 = (0,0,1)\}$

$$T(e_2) = 1\cdot\alpha_1 + (-1)\cdot\alpha_2 + 7\cdot\alpha_3$$
$$= 1\cdot(1,0,0) + (-1)\cdot(0,1,0) + 7\cdot(0,0,1)$$
$$= (1, -1, 7)$$

$$T : V_2 \to V_3$$

$$T(x, y) = T(xe_1 + ye_2)$$
$$= x\cdot T(e_1) + y\cdot T(e_2)$$
$$= x\cdot(1,2,0) + y\cdot(1,-1,7)$$
$$= (x+y, 2x-y, 7y)$$

$$\to [T]_{\beta_1, \beta_2} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}$$

(Matrix to L.T.)

$$\beta_1 = \{e_1, e_2\}$$
$$\beta_2 = \{\alpha_1, \alpha_2, \alpha_3\}$$

$$T(e_1) = 1\cdot\alpha_1 + 2\cdot\alpha_2 + 0\cdot\alpha_3$$
$$= 1\cdot(1,0,0) + 2\cdot(0,1,0) + 0\cdot(0,0,1)$$
$$= (1, 2, 0).$$

$\rightarrow \beta = \{e_1, e_2, e_3\}$.

$[T]_\beta$

$T(e_1) = 0\cdot e_1 + 1\cdot e_2 - 1\cdot e_3$

$= 0\cdot(1,0,0) + 1\cdot(0,1,0) - 1\,(0,0,1)$

$= (0, 1, -1)$

$T(e_2) = 1\cdot e_1 + 0\cdot e_2 - e_3$

$= (1,0,0) + (0,0,0) + (0,0,-1)$

$= (1, 0, -1)$

Prob: If the matrix of a linear operator T on $V_3(\mathbb{R})$ w.r.t the standard basis is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

then describe $T: V_3 \to V_3$ explicitly. Also find matrix of T w.r.t. the basis $\{(0,1,-1), (1,-1,1), (-1,1,0)\}$.

$$= (y+z,\ x-z,\ -x-y)$$

Let $B_1 = \{ \underset{\downarrow\,\alpha_1}{(0,1,-1)},\ \underset{\downarrow\,\alpha_2}{(1,-1,1)},\ \underset{\downarrow\,\alpha_3}{(-1,1,0)} \}$

$$T(\alpha_1) = T(0,1,-1) = (0,\ 1,\ -1)$$
$$= 1\cdot\alpha_1 + 0\cdot\alpha_2 + 0\cdot\alpha_3$$

$$T(\alpha_2) = T(1,-1,1) = (0,\ 0,\ 0)$$
$$= 0\cdot\alpha_1 + 0\cdot\alpha_2 + 0\cdot\alpha_3$$

$$T(\alpha_3) = T(-1,1,0) = (1,-1,\ 0)$$
$$= 0\cdot\alpha_1 + 0\cdot\alpha_2 - \alpha_3$$

$$T(e_3) = e_1 - e_2 + 0\cdot e_3$$
$$= (1,0,0) - (0,1,0) + (0,0,0)$$
$$= (1,-1,0)$$

$$T(x,y,z) = T(x e_1 + y e_2 + z e_3)$$
$$= x\,T(e_1) + y\,T(e_2) + z\cdot T(e_3)$$
$$= x(0,\ 1,-1) + y(1,\ 0,-1) + z\,(1,-1,0)$$

to the standard bases

$B_1 = \{1, x, x^2, x^3\}$

$B_2 = \{1, x, x^2\}$.

$\longrightarrow [T]_{\beta_1, B_2}$

$D(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$D(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2$.

$$[T]_{\beta_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{3\times 3}$$

Prob: Let $D : P_3 \to P_2$ be the polynomial differential transformation $D(f(x)) = \frac{d}{dx}(f(x))$. Find the matrix of D relative

$$[T]_{B_1, B_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3 \times 4}$$