KGS IAS

Maths Optional

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Let $S = \{v_1, v_2, \dots v_n\}$ is a basis of V.

Let $T = \{w_1, w_2, \dots w_m\}$ be a subset of V.

suppose $m > n$

we will prove that T is L.D.

Th: A FDVS $V(F)$ has dimension n iff n is the max. no. of Linearly independent vectors in any subset of V.

Proof: suppose $\dim V = n$.

To prove: n is the max. no. of L.I. vectors in any subset of V.

$$\Rightarrow x_1(\alpha_{11} v_1 + \alpha_{21} v_2 + \cdots + \alpha_{n1} v_n) +$$
$$x_2(\alpha_{12} v_1 + \alpha_{22} v_2 + \cdots + \alpha_{n2} v_n) +$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$x_m(\alpha_{1m} v_1 + \alpha_{2m} v_2 + \cdots + \alpha_{nm} v_n) = 0$$

$$\Rightarrow (\alpha_{11} x_1 + \alpha_{12} x_2 + \cdots + \alpha_{1m} x_m) v_1 +$$
$$(\alpha_{21} x_1 + \alpha_{22} x_2 + \cdots + \alpha_{2m} x_m) v_2 +$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$(\alpha_{n1} x_1 + \alpha_{n2} x_2 + \cdots + \alpha_{nm} x_m) v_n = 0$$

$w_i \in T \subseteq V$

$\Rightarrow w_i \in V$

and since S is a basis of V.

$\therefore \exists\, \alpha_{ji} \in F$ s.t

$$w_i = \alpha_{1i} v_1 + \alpha_{2i} v_2 + \cdots + \alpha_{ni} v_n$$
$$\forall\, i = 1, 2, \ldots m$$

Consider

$$x_1 w_1 + x_2 w_2 + \cdots + x_m w_m = 0 \quad \text{—} (1)$$
$$x_i \in F.$$

This system of eqns. has non-zero soln.

$\therefore$ Some of x_i's must be non-zero.

(1) $\Rightarrow$ $T = \{w_1, w_2, \ldots w_m\}$ is L.D.

$\therefore$ n is the max. number of LI vectors in any subset of V.

$\therefore$ S is L.I

$\therefore$

$$\alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1m}x_m = 0$$
$$\alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2m}x_m = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\alpha_{n1}x_1 + \alpha_{n2}x_2 + \cdots + \alpha_{nm}x_m = 0$$

No of variables $= m$

No. of eqns. $= n$

As $m > n$

clearly $L(S) \subseteq V$.

Let $v \in V$

$\therefore \{v, v_1, v_2, \ldots v_n\}$ is

a L.D. subset of V.

[$\because$ this set has $(n+1)$ elements]

$\therefore \exists\, \alpha, \alpha_1, \alpha_2, \ldots \alpha_n$ not all zeros s.t

$$\alpha v + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \quad \text{--- (1)}$$

Conversely, suppose that n is the max. number of L.I. vectors in any subset of V.

To prove: $\dim V = n$

Let $S = \{v_1, v_2, \ldots v_n\}$ be a L.I. subset of V.

To prove S as a basis of V we have to prove $L(S) = V$

$\therefore$ our supposition is wrong.

and so $\alpha \neq 0$

$\therefore \alpha^{-1}$ exists

(11)

$$\alpha^{-1}(\alpha u + \alpha_1 u_1 + \dots + \alpha_n u_n) = \alpha^{-1} \cdot 0 = 0$$

$$\Rightarrow \underbrace{(\alpha^{-1}\alpha)}_{1} u + (\alpha^{-1}\alpha_1) u_1 + \dots + (\alpha^{-1}\alpha_n) u_n = 0$$

$$\Rightarrow u = -\left[(\alpha^{-1}\alpha_1) u_1 + (\alpha^{-1}\alpha_2) u_2 + \dots + (\alpha^{-1}\alpha_n) u_n\right]$$

If $\alpha = 0$,

(11) $\Rightarrow$

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_n = 0$$

[As S is L.I.]

$$\therefore \alpha = \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

Which is a contradiction to the our hypotheses that some α or α_i's are non-zero.

Th: If $\dim V = n$ then any $n+1$ vectors are L.D.

Proof: result follows from the 1st part of the above th.

$\therefore v \in L(S)$

$\therefore V \subseteq L(S)$

So $L(S) = V$

$\therefore S$ is a basis of V.

$\therefore \dim V = n$.

s.t. $\{v_1, v_2, \ldots v_r, v_{r+1}, \ldots v_n\}$ is a bases of V.

proof: Let $\dim V = n$, n is the max. no. of L.I vectors in any subset of V.

If $\{v_1, v_2, — v_r\}$ spans V and $r = n$ then it is a bases of V. And there is nothing to prove.

Extension theorem: Every finite linearly independent subset of a FDVS V over F can be extended to form the bases of $V(F)$.

or

If V is a FDVS over F and if $v_1, v_2, — v_r$ are L.I vectors in V, then there exist vectors $v_{r+1}, v_{r+2}, \ldots v_n \in V$

Clearly, $L(S) \subseteq V$

Let $v \in V$

$\therefore \{v, v_1, v_2, \ldots v_n\}$ is L.D. [It has n+1 vectors]

$\therefore \exists\, \alpha, \alpha_1, \ldots \alpha_n$ (not all zeros) s.t

$$\alpha v + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \quad (1)$$

If $\{v_1, v_2, \ldots v_r\}$ doesn't span V, then it is not a basis of V.

Let $S = \{v_1, v_2, \ldots v_r, v_{r+1}, v_{r+2}, \ldots v_n\}$ be the maximal L.I subset of V.

To prove S is a basis of V.

i.e. to prove $L(S) = V$.

$\therefore$ our supposition is wrong.

and so $\alpha \neq 0$

$\therefore \alpha^{-1}$ exists

(1) $\Rightarrow$

$$\alpha^{-1}(\alpha u + \alpha_1 u_1 + \cdots + \alpha_n u_n) = \alpha^{-1} \cdot 0 = 0$$

$$u = -\left[(\alpha^{-1} \cdot \alpha_1) u_1 + \cdots (\alpha^{-1} \alpha_n) u_n\right]$$

$\therefore u \in L(S)$

If $\alpha = 0$, then

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

[$\because S$ is L.I.]

i.e. $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

which is a contradiction to the hypothesis that some of α or α_i are non-zero.

Th: If $\dim V = n$ and $\{v_1, v_2, \ldots v_m\}$ is a L.I. subset of V then $m \leq n$.

or

If $\dim V = n$ then a L I subset of V cannot have more than n elements.

Proof: Result follows from the fact that if $\dim V = n$

$\therefore V \subseteq L(S)$

and so

$L(S) = V$

$\therefore$ S is a bases of V.

$\therefore$ It can be extended to form the basis of V.

But $\dim V = n$

and S has n elements

$\therefore$ S is a basis of V.

then n is the max. no. of L.I. vectors in any subset of V.

Th: If $\dim V = n$ and $S = \{v_1, v_2, \dots v_n\}$ is a L.I. subset of V then S is a basis of V.

Proof: $\because$ S is L.I.

and S has n elements.

$\therefore S$ is a basis of V.

Th: If $\dim V = n$ and $S = \{v_1, v_2, \ldots, v_n\}$ spans V then S is a basis of V.

proof: $\because L(S) = V$

$\therefore \exists$ a subset of S which will be a basis of V.

But $\dim V = n$

$\therefore$ Any basis of V can not have fewer than n elements

Th: Let $S = \{v_1, v_2, \ldots v_n\}$ be a basis of a FDVS $V(F)$ of dimension n. Then any element $v \in V$ can be uniquely expressed as

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

$$a_i \in F.$$

proof: $\because S = \{v_1, v_2, \ldots v_n\}$ is a basis of V.

$\therefore L(S) = V$.

Let $v \in V = L(S)$

$\therefore \exists$ scalars $a_1, a_2, \ldots a_n \in F$.

s.t.

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \quad (i)$$

Let, if possible

$$v = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n \quad (ii)$$

$$b_i \in F.$$

$\therefore$ Expression in (I) is unique.

From (I) & (II)

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$$

$$\Rightarrow (a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \cdots + (a_n - b_n) v_n = 0$$

$\Rightarrow a_1 - b_1 = 0$ i.e. $a_1 = b_1$ [As S is L.I]

$a_2 - b_2 = 0$ i.e. $a_2 = b_2$

$\vdots$

$a_n - b_n = 0$ i.e. $a_n = b_n$

$\rightarrow L(S) = \mathbb{C}^2(\mathbb{R})$.

clearly, $L(S) \subseteq \mathbb{C}^2(\mathbb{R})$

let $(a+ib, c+id) \in \mathbb{C}^2$.

observe that

$(a+ib, c+id)$

$= a(1,0) + b(i,0)$

$+ c(0,1) + d(0,i)$

L.C. of elements of S

$\therefore (a+ib, c+id) \in L(S)$.

<u>prob:</u> Show that the set

$S = \{(1,0), (i,0), (0,1), (0,i)\}$

is a basis for $\mathbb{C}^2(\mathbb{R})$.

$\rightarrow$ <u>S is L.I.</u>

$a(1,0) + b(i,0) + c(0,1) + d(0,i)$

$= (0,0)$

$\Rightarrow (a+bi, c+id) = (0,0)$

$\Rightarrow a+ib = 0+i0, \; c+id = 0+i\cdot 0$

$\Rightarrow a=0, b=0, c=0, d=0$

prob: Let $\{a, b, c\}$ be a basis for $\mathbb{R}^3$. prove that the set $\{a+b, b+c, c+a\}$ is also a basis of $\mathbb{R}^3$.

Sol^n: $\because \{a, b, c\}$ is a basis of $\mathbb{R}^3$.

$\therefore$ a, b, c are L.I.

Now consider

$x(a+b) + y(b+c) + z(c+a) = 0$

$x, y, z \in \mathbb{R}$

$\therefore C^2(\mathbb{R}) \subseteq L(S)$

So $L(S) = C^2(\mathbb{R})$

$\therefore S$ is a basis of $C^2(\mathbb{R})$.

Similarly, we can find

$y = 0,\ z = 0$.

$\therefore \{a+b, b+c, c+a\}$ is

L.I. and it is a

basis of $\mathbb{R}^3$.

as $\dim \mathbb{R}^3 = 3$.

$\Rightarrow (x+z)a + (x+y)b + (y+z)c = 0$

$\Rightarrow \quad x+z = 0$

$\quad\quad x+y = 0$

$\quad\quad \underline{y+z = 0}$

[As a, b, c are L.I.]

$2(x+y+z) = 0$

$\Rightarrow x+y+z = 0$

$\quad\quad -\ y+z = 0$

$x = 0$

$$(a,b,c) = x_1(1,0,0) + x_2(1,1,0) + x_3(1,1,1) + x_4(0,1,0)$$

$x_i \in \mathbb{R}$

$$\Rightarrow \quad x_1 + x_2 + x_3 = a$$

$$x_2 + x_3 + x_4 = b$$

$$\underline{x_3 = c}$$

Let $x_4 = \underline{-c}$

$$x_2 = \underline{b}$$

$$x_1 = \underline{a - b - c}$$

prob: Show that the set

$$S = \{(1,0,0), (1,1,0), (1,1,1), (0,1,0)\}$$

is a spanning set of $\mathbb{R}^3$ but is not a basis of $\mathbb{R}^3$.

Sol^n: To prove: $L(S) = \mathbb{R}^3$.

Clearly, $L(S) \subseteq \mathbb{R}^3$.

Let $(a,b,c) \in \mathbb{R}$

$\Rightarrow (a,b,c) \in L(S)$

$\therefore \mathbb{R}^3 \subseteq L(S)$

& so $\underline{\mathbb{R}^3 = L(S)}$.

$\therefore \text{Dim}\,\mathbb{R}^3 = 3$

and S has 4 elements.

$\therefore$ S is not a basis of $\mathbb{R}^3$.

$L(S)$ is a subspace of $\mathbb{R}^4$.

S is LD.

Observe that

$(1,2,0,1) = 1\cdot(1,0,0,0) + 2(0,1,0,0) + 1\cdot(0,0,0,1)$

$v_3 = v_1 + 2v_2 + v_4$.

Let $S' = \{v_1, v_2, v_4\}$.

$\therefore \underline{L(S) = L(S')}$ (prove it!)

Prob: Find the dimension of the sub-space of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$. Hence find a basis for the subspace.

Sol^n: Let $S = \{\underset{v_1}{(1,0,0,0)}, \underset{v_2}{(0,1,0,0)}, \underset{v_3}{(1,2,0,1)}, \underset{v_4}{(0,0,0,1)}\}$.

Thus S' is a basis of

$L(S)$.

$\text{Dim } L(S) = 3$.

S' is L.I.

Let $a e_1 + b e_2 + c e_4 = 0$

$a, b, c \in \mathbb{R}$

$$a(1,0,0,0) + b(0,1,0,0) + c\cdot(0,0,0,1) = (0,0,0,0)$$

$\Rightarrow a = 0, \; b = 0, \; c = 0$

$\therefore S'$ is L.I.

$$L(S) = \{a(1,1,1)+b(1,0,0) : a, b \in \mathbb{R}\}$$

$$= \{(a+b, a, a) : a, b \in \mathbb{R}\}$$

$$(0,1,0) \notin L(S)$$

$\therefore S' = \{(1,1,1), (1,0,0), (0,1,0)\}$ is L.I.

$\therefore S'$ is a basis of $\mathbb{R}^3$.

As $\dim \mathbb{R}^3 = 3$

<u>prob:</u> Extend the set $\{(1,1,1), (1,0,0)\}$ to form a basis of $\mathbb{R}^3$.

<u>Sol^n:</u> Let $S = \{(1,1,1), (1,0,0)\}$

<u>S is L.I.</u>

$$a(1,1,1)+b(1,0,0) = (0,0,0)$$

$$\Rightarrow a+b=0$$

$$\underline{a=0} \quad \therefore b=0$$