Maths Optional

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Prob: Extend the set $S = \{(1,1,0)\}$ to form two different bases of $\mathbb{R}^3$.

Solⁿ. S is L.I.

As $\alpha(1,1,0) = (0,0,0)$

$\Rightarrow \alpha = 0$

$L(S) = \{a(1,1,0) : a \in \mathbb{R}\}$

$= \{(a,a,0) : a \in \mathbb{R}\}$

$(0,0,1) \notin L(S)$

$\therefore S_1 = \{(1,1,0), (0,0,1)\}$ is L.I.

$L(S_1) = \{a(1,1,0) + b(0,0,1) : a, b \in \mathbb{R}\}$

Also $(0,1,0) \notin L(S_1)$

$\therefore S_3 = \{(1,1,0), (0,0,1), (0,1,0)\}$ is

L.I.

$\text{Dim } \mathbb{R}^3 = 3$

and S_3 has three vectors.

$\therefore S_3$ is a basis of $\mathbb{R}^3$.

So $L(S_1) = \{(a,a,b) : a, b \in \mathbb{R}\}$

$(0,1,1) \notin L(S_1)$

$\therefore S_2 = \{(1,1,0), (0,0,1), (0,1,1)\}$

is L.I.

$\text{Dim } \mathbb{R}^3 = 3$

and S_2 has 3 L.I. vectors.

$\therefore S_2$ is a basis of $\mathbb{R}^3$

So the set can not be extended to form the basis of $\mathbb{R}^4$.

Prob: Can the set

$\{(1,0,0,0), (0,1,0,0), (1,-1,0,0)\}$

be extended to form a basis of $\mathbb{R}^4$?

Sol^n: The given set is L.D.

As

$$(1,-1,0,0) = 1\cdot(1,0,0,0) - 1\cdot(0,1,0,0)$$

$$L(S) = \{(a, -b, a+b, 0) : a, b \in \mathbb{R}\}$$

$$(0, 0, 0, 1) \notin L(S).$$

$$\therefore S_1 = \{(1,0,1,0), (0,-1,1,0), (0,0,0,1)\} \text{ is L.I.}$$

$$L(S_1) = \{a(1,0,1,0) + b(0,-1,1,0) + c(0,0,0,1) : a, b, c \in \mathbb{R}\}$$

Prob: Given two linearly independent vectors $(1,0,1,0)$ and $(0,-1,1,0)$ of $\mathbb{R}^4$, find a basis of $\mathbb{R}^4$ which includes these two vectors.

Soln: Let $S = \{(1,0,1,0), (0,-1,1,0)\}$

S is L.I. (Given)

$$L(S) = \{a(1,0,1,0) + b(0,-1,1,0) : a, b \in \mathbb{R}\}$$

Prob: Let V be the vector space of all 2×2 symmetric matrices over $\mathbb{R}$. Find a basis and the dimension of V.

Soln:

$$V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$L(S_1) = \{ (a, -b, a+b, c) : a, b, c \in \mathbb{R} \}$$

$$(0, 0, 1, 0) \notin L(S_1)$$

Now $S_2 = \{ (1, 0, 1, 0), (0, -1, 1, 0), (0, 0, 0, 1), (0, 0, 1, 0) \}$

is L.I.

$\operatorname{Dim} \mathbb{R}^4 = 4$

$\therefore S_2$ is a basis of $\mathbb{R}^4$.

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\implies a = b = c = 0$$

clearly, $L(S) \subseteq V$

let $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in V$

Consider

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$S \subseteq V$$

<u>S is L.I</u>

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ S is a basis of V.

Dim V = 3

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Lin. comb. of element of S.

$$\therefore \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in L(S)$$

$$\therefore V \subseteq L(S)$$

So $V = L(S)$.

Prob: Let V be the vector space of 3×3 symmetric matrices over $\mathbb{R}$. Then show that dimension of V is 6 by establishing a basis for V.

Sol^n:

$$V = \left\{ \begin{bmatrix} a & f & g \\ f & b & h \\ g & h & c \end{bmatrix} : a, b, c, f, g, h \in \mathbb{R} \right\}$$

Consider

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$\longrightarrow$ S is L.I.

$\longrightarrow$ $L(S) = V$

$\left\{ \text{prove it} \right\}$

$\therefore \operatorname{Dim} V = 6.$

$$\text{Dim } V = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Note: ① V – VS of 2×2 symmetric matrices over $\mathbb{R}$

$$\text{Dim } V = 3 = 1+2$$

② V – vector space of 3×3 symmetric matrices over $\mathbb{R}$.

$$\text{Dim } V = 6 = 1+2+3$$

③ V – vector space of n×n Symmetric matrices over $\mathbb{R}$.

prob: V be the vector space of all 2×2 skew-symmetric matrices over the field F. Show that dim V = 1 by establishing a basis for V.

Sol^n: $V = \left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} : a \in F \right\}$

Consider $S = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

S is L.I.:

$$a \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad a \in F.$$

$$\Rightarrow \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow a = 0.$$

Clearly, $L(S) \subseteq V$

Let $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \in V$

prob: Let V be the vector space of 3×3 skew-symmetric matrices over the field F. Show that dim V = 3 by establishing a basis for V.

Solⁿ: $V = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} : a, b, c \in F \right\}$

$S = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$

$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$

$\therefore \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \in L(S)$

$\therefore V \subseteq L(S)$

So $V = L(S)$

$\therefore$ S is a basis of V.

$\text{Dim } V = 1$

(II) Let V be the vector space of 3×3 skew-symmetric matrices over the field F.

$$\text{Dim}\, V = 3 = \underline{2+1}$$

$$= (3-1)+(3-2)$$

(III) Let V be the vector-space of $n\times n$ skew-symmetric matrices over the field F.

$$\text{Dim}\, V = (n-1)+(n-2)+(n-3)+ \cdots + 3+2+1$$

$$= \frac{(n-1)\times n}{2}.$$

S is L.I. $\Big\}$ prove it.
$L(S)=V$ $\Big\}$ (HW)

$\therefore$ S is a basis of V.

$$\text{Dim}\, V = 3$$

Note (1) V be the vector space of 2×2 skew-symmetric matrices over F.

$$\text{Dim}\, V = 1$$

$$= 2-1$$

$E_{ij} \to (i,j)^{th}$ elt. $= 1$.

Note:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}_{3\times 2} : a, b, c, d, e, f \in F \right\}.$$

$$S = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{12}}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{21}}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{2,2}}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{3,1}}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{33}} \right\}$$

prob: Let V be the set of all real valued functions $y = f(x)$ satisfying $\frac{d^2y}{dx^2} + 4y = 0$.
Prove that V is 2-dimensional real vector space.

Soln: $\frac{d^2y}{dx^2} + 4y = 0$ — (1)

Aux. eqn.: $m^2 + 4 = 0$
$\Rightarrow m = \pm 2i$

Let V be the vector space of $m \times n$ matrices over the field F.
Let $E_{ij} \in V$ be the matrix with (i,j)th element as '1' and other elements as '0'.
The the set $\{E_{ij}\}$ will be a basis of V.

$\text{Dim } V = m \cdot n$

This basis — standard basis of V.

Wronskian $(W(x))$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$
$$= 2\left[\cos^2 2x + \sin^2 2x\right]$$
$$= 2 \times 1 = 2 \neq 0$$

$y = a\cos 2x + b\sin 2x$, $a, b \in \mathbb{R}$

y is a sol^n of (1)

$V = \left\{ y = a\cos 2x + b\sin 2x : a, b \in \mathbb{R} \right\}$

Easily, it can be shown that V is a vector space over $\mathbb{R}$.

Consider $S = \{ \underset{\downarrow}{\underset{y_1}{\cos 2x}}, \underset{y_2}{\sin 2x} \} \subseteq V$

So, $V = L(S)$

$\therefore S$ is a basis of V.

$\text{Dim } V = 2$

$\therefore S$ is L.I.

clearly $L(S) \subseteq V$

$y = a\cos 2x + b\sin 2x \in V$

$y = a \cdot \cos 2x + b \cdot \sin 2x$ $\quad a, b \in \mathbb{R}$

Lin. comb. of elements of S.

$\therefore y \in L(S)$

$\therefore V \subseteq L(S)$

Aux. eqn:

$$m^3-7m-6=0$$

$$\Rightarrow (m+1)\left[m^2-m-6\right]=0$$

$$\Rightarrow (m+1)\left[m^2-3m+2m-6\right]=0$$

$$\Rightarrow (m+1)(m+2)(m-3)=0$$

$$\Rightarrow m=-1,-2,3$$

$\therefore$ Soln. of (1) is given by

$$y=c_1\cdot e^{-x}+c_2\cdot e^{-2x}+c_3\cdot e^{3x}$$

Prob: Let V be the set of all real valued functions $y=f(x)$ satisfying $\frac{d^3y}{dx^3}-7\frac{dy}{dx}-6y=0$.

Show that V is a 3-dimensional real vector space. Write down a basis of this vector space.

Soln:

$$\frac{d^3y}{dx^3}-7\frac{dy}{dx}-6y=0 \quad \text{—— (1)}$$

Sol:-

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Wronskian

$$= \begin{vmatrix} e^{-x} & e^{-2x} & e^{3x} \\ -e^{-x} & -2e^{-2x} & 3e^{3x} \\ e^{-x} & 4e^{-2x} & 9e^{3x} \end{vmatrix}$$

$$= e^{-x} \cdot e^{-2x} \cdot e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$V = \{ y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} : c_1, c_2, c_3 \in \mathbb{R} \}$

It can be easily shown that V is a vector space over $\mathbb{R}$.

Consider $S = \{ \underset{y_1}{e^{-x}}, \underset{y_2}{e^{-2x}}, \underset{y_3}{e^{3x}} \} \subseteq V$

$\therefore S$ is L.I.

$L(S) \subseteq V$ (obvious)

let $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} \in V$

$$y = \underbrace{c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}}_{}$$

$c_1, c_2, c_3 \in \mathbb{R}$

L.C. of elements of S

$\therefore y \in L(S)$

$\therefore V \subseteq L(S)$

So $V = L(S)$

$\therefore S$ is a basis of V

$\boxed{\text{Dim } V = 3}$

$$= e^{0x} \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 4 \\ 0 & 3 & 8 \end{vmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= 1 \cdot \begin{vmatrix} -1 & 4 \\ 3 & 8 \end{vmatrix} = -8 - 12$$

$$= -20 \neq 0$$

$W(x) \neq 0$

S is L.I.

$$a\cdot 1+b\cdot\sqrt{2}=0 \qquad a,b\in\mathbb{Q}$$

$$=0\cdot 1+0\cdot\sqrt{2}$$

$$\Rightarrow a=0,\ b=0$$

Clearly, $L(S)\subseteq\mathbb{Q}(\sqrt{2})$

$$a+b\sqrt{2}\in\mathbb{Q}(\sqrt{2})$$

$$a+b\sqrt{2}=a\cdot 1+b\cdot\sqrt{2},\ a,b\in\mathbb{Q}.$$

L.C. of S.

Prob: Show that the dimension of the vector space $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ is 2.

Soln:

$$\mathbb{Q}(\sqrt{2})=\{a+b\cdot\sqrt{2} : a,b\in\mathbb{Q}\}$$

Consider

$$S=\{1,\sqrt{2}\}$$

prob: Show that the dimension of the vector space $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$ is 4.

Soln:

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\cdot\sqrt{3} : a, b, c, d \in \mathbb{Q}\}$$

$S = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$

HW

$\therefore a + b\sqrt{2} \in L(S)$

$\therefore \mathbb{Q}(\sqrt{2}) \subseteq L(S)$

So $L(S) = \mathbb{Q}(\sqrt{2})$

$\therefore S$ is a basis of $\mathbb{Q}(\sqrt{2})$

$\text{Dim}\, \mathbb{Q}(\sqrt{2}) = 2$.