Maths Optional

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$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 0$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ (1) is not exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2y}{2y} = 1$$

$\therefore IF = e^{\int 1 dx} = e^{x}$

Rule 3: $M dx + N dy = 0$

Non-exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x) \text{ or } K.$$

$$IF = e^{\int f(x) dx} \text{ or } e^{\int K dx}$$

Prob: $(x^2 + y^2 + 2x) dx + 2y dy = 0$ — (1)

$M dx + N dy = 0$

$M = x^2 + y^2 + 2x$

$N = 2y$

2nd Method:

$$(x^2+y^2+2x)\,dx+2y\,dy=0$$

$$\Rightarrow (x^2+y^2)\,dx+2x\,dx+2y\,dy=0$$

$$\Rightarrow (x^2+y^2)\,dx+d(x^2)+d(y^2)=0$$

$$\Rightarrow (x^2+y^2)\,dx+d(x^2+y^2)=0$$

$$\Rightarrow dx+\frac{1}{(x^2+y^2)}\,d(x^2+y^2)=0$$

Int

$$x+\log(x^2+y^2)=\text{const}$$

multiplying ① with e^x, we have

$$e^x(x^2+y^2+2x)\,dx+2\underline{e^x y}\,dy=0$$

Soln. is

$$\int x^2e^x\,dx+y^2\int e^x\,dx+2\int xe^x\,dx=\text{const}$$

$$\Rightarrow x^2\int e^x\,dx-\cancel{2\int xe^x\,dx}+y^2e^x+\cancel{2\int xe^x\,dx}=\text{const}$$

$$\Rightarrow x^2e^x+y^2e^x=\text{const}$$

$$\Rightarrow e^x(x^2+y^2)=c$$

Rule 4: $M\,dx + N\,dy = 0$

Not exact.

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y) \text{ or } K$$

$$I.F. = e^{\int f(y)\,dy} \text{ or } e^{\int K\,dy}$$

Prob: $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ — (1)

$$\log(x^2+y^2) = \text{const} - x$$

$$\Rightarrow x^2 + y^2 = e^{\text{const} - x} = e^{\text{const}} \cdot e^{-x}$$

$$\Rightarrow x^2 + y^2 = \frac{c}{e^x}$$

$$\Rightarrow e^x(x^2+y^2) = c$$

$$= \frac{1}{y(\cancel{y^3+2})}\,(-3)\,\cancel{(-3y^3-6)} = -\frac{3}{y}$$

$$IF = e^{-3\int \frac{1}{y}dy}$$

$$= e^{-3\log y}$$

$$= e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

Multiplying ① with $\frac{1}{y^3}$, we have

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

$$M = y^4 + 2y$$

$$N = xy^3 + 2y^4 - 4x$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2$$

$$\frac{\partial N}{\partial x} = y^3 - 4$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(y^3+2)}\left(y^3 - 4 - 4y^3 - 2\right)$$

$IF = x^h \cdot y^K$ — h and K are to be determined.

Multiplying (1) with $x^h \cdot y^K$

$M_1 dx + N_1 dy = 0$

To be exact

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

h, K determine

Soln. is

$$\left(y + \frac{2}{y^2}\right)x + y^2 = \text{const.}$$

$$\Rightarrow xy + \frac{2x}{y^2} + y^2 = C.$$

Rule 5: $M dx + N dy = 0$

Not exact

$$x^\alpha y^\beta (my\,dx + nx\,dy) + x^{\alpha_1} \cdot y^{\beta_1} (m_1 y\,dx + n_1 x\,dy) = 0 \quad \text{—(1)}$$

$$M_1\,dx + N_1\,dy = 0$$

$$M_1 = 4x^{h+1}\cdot y^{k+1} + 3x^{h}\cdot y^{4+k}$$

$$N_1 = 2x^{h+2}\cdot y^{k} + 5x^{h+1}\cdot y^{k+3}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$4(k+1)\,x^{h+1}\cdot y^{k} + 3(k+4)\cdot x^{h}\cdot y^{k+3}$$

$$= 2(h+2)\cdot x^{h+1}\cdot y^{k} + 5(h+1)x^{h}\cdot y^{k+3}$$

Prob: Solve

$$x\cdot(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0 \quad (1)$$

Multiplying (1) with $x^{h}\cdot y^{k}$ (h, k are to determined s.t (1) becomes exact)

$$x^{h}\cdot y^{k}\left[(4xy + 3y^4)\,dx + (2x^2 + 5xy^3)\,dy\right] = 0$$

$$(4x^{h+1}\cdot y^{k+1} + 3x^{h}y^{4+k})\,dx + (2x^{2+h}\cdot y^{k} + 5x^{1+h}\cdot y^{3+k})\,dy = 0 \quad (11)$$

Put $h=2$ & $k=1$ in (1),

$$\left(4x^3y^2+3x^2y^5\right)dx+\left(2x^4y^1+5x^3y^4\right)dy=0$$

Sol^n. is

$$\boxed{x^4y^2+x^3y^5=c}$$

$\therefore 4(k+1)=2(h+2)$

$\Rightarrow 4k+\cancel{4}=2h+\cancel{4}$

$\Rightarrow 4k-2h=0 \Rightarrow 2k-h=0$

$3(k+4)=5(h+1)$

$\Rightarrow 3k+12=5h+5$

$\Rightarrow 3k+12=10k+5$

$\Rightarrow 7=7k$

$k=1$

$h=2$

$$\frac{ds}{dt} + P \cdot s = Q.$$

P & Q are fn of t.

Linear in s.

Linear diff. eqn:

$\rightarrow$ $$\frac{dy}{dx} + Py = Q.$$

P and Q are functions of x.

$\rightarrow$ $$\frac{dx}{dy} + Px = Q$$

P & Q are functions of y.

$\rightarrow \dfrac{dx}{dy} + Px = Q.$

P & Q — fn. of 'y'.

$I.F. = e^{\int P\,dy}.$

Solⁿ is

$x \cdot I.F. = \int (I.F.)\,Q\,dy + C$

$\rightarrow$

$\dfrac{dy}{dx} + Py = Q$

Linear in y.

$I.F. = e^{\int P\,dx}$

Solⁿ is

$y \cdot I.F. = \int (I.F.)\,Q\,dx + C.$

Solⁿ is

$y \times \sin x = \int \sin x \cdot 2\cos x \, dx + \text{const.}$

$= \int \sin 2x \, dx + \text{const.}$

$= -\dfrac{\cos 2x}{2}$

$\Rightarrow \boxed{2y\sin x + \cos 2x = C}$

Prob:

$\dfrac{dy}{dx} + y\cot x = 2\cos x$

Comparing with.

$\dfrac{dy}{dx} + Py = Q.$

$P = \cot x, \quad Q = 2\cos x$

$If = e^{\int \cot x \, dx}$

$= e^{\int \frac{\cos x}{\sin x} dx} = e^{\log \sin x}$

$= \sin x$

Sol^n is

$$y \times \frac{1}{x^2} = \int \frac{1}{x^{\cancel{2}}} \times \cancel{x} \, dx + \text{const}$$

$$\boxed{\frac{y}{x^2} = \log x + C}$$

Prob: $x\frac{dy}{dx} - 2y = x^2$

$$\Rightarrow \frac{dy}{dx} - \frac{2}{x} \times y = x$$

$$P = -\frac{2}{x}, \quad Q = x$$

$$I.F. = e^{-\int \frac{2}{x} dx}$$

$$= e^{-2 \log x}$$

$$= e^{\log x^{-2}}$$

$$= \frac{1}{x^2}.$$

Solⁿ. is

$$y \times \log x = 2\int (\log x)\cdot \frac{1}{x}\,dx + \text{const}$$

$$= 2\int z\,dz + \text{const}$$

(put $z = \log x$)

$$y \log x = z^2 + \text{const}$$

$$y(\log x) = (\log x)^2 + c$$

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prob:

$$x \log x \frac{dy}{dx} + y = 2\log x$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x\log x}\,y = \frac{2}{x}$$

$$P = \frac{1}{x\log x},\quad Q = \frac{2}{x}$$

$$\text{I}f = e^{\int \frac{1}{x\log x}dx}$$

$$= e^{\int \frac{1}{t}dt}, \quad t = \log x$$

$$= e^{\log t} = t = \log x.$$

$$IF = e^{-\int \frac{1}{2\sin x \cdot \cos x} dx}$$

$$= e^{-\frac{1}{2}\int \frac{1}{\frac{\sin x}{\cos x} \cdot \cos^2 x} dx}$$

$$= e^{-\frac{1}{2}\int \frac{\sec^2 x}{\tan x} dx}$$

$$= e^{-\frac{1}{2}\log \tan x}$$

$$= e^{\log \frac{1}{\sqrt{\tan x}}}$$

$$= \frac{1}{\sqrt{\tan x}}$$

Sol^n is

$$y \cdot \frac{1}{\sqrt{\tan x}} = \frac{1}{2}\int \frac{\sec^2 x}{\sqrt{\tan x}} dx + \text{const}$$

Prob: $\sin 2x \frac{dy}{dx} = y + \tan x$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{\sin 2x} + \frac{\tan x}{\sin 2x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{\sin 2x} \cdot y = \frac{\frac{\cancel{\sin x}}{\cos x}}{2\cancel{\sin x} \cdot \cos x}$$

$$= \frac{1}{2}\sec^2 x.$$

$$P = -\frac{1}{\sin 2x}, \quad Q = \frac{1}{2}\sec^2 x$$

prob: $x\cos x \frac{dy}{dx} + (x\sin x + \cos x)y = 1$

HW

Ans: $y \cdot x\sec x = \tan x + C$

$$\frac{y}{\sqrt{\tan x}} = \frac{1}{2}\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx + \text{const.}$$

$$= \frac{1}{2}\int t^{-1/2}\,dt + \text{const}$$

$$= \frac{1}{2} \times \frac{t^{1/2}}{\cancel{1/2}} + \text{const}$$

put $t = \tan x$ in the integral

$$\frac{y}{\sqrt{\tan x}} = \sqrt{\tan x} + C$$

$$\boxed{y = \tan x + C \cdot \sqrt{\tan x}}$$

Prob: $(x+2y^3)\dfrac{dy}{dx} = y$

$\Rightarrow (x+2y^3) = y\cdot\dfrac{dx}{dy}$

$\Rightarrow \dfrac{1}{y}\cdot x + 2y^2 = \dfrac{dx}{dy}$

$\Rightarrow \dfrac{dx}{dy} - \dfrac{1}{y}x = 2y^2$ —— ①

$\dfrac{dx}{dy} + Px = Q$ ——

① is linear in 'x'.

$P = -\dfrac{1}{y}, \quad Q = 2y^2$

$IF = e^{\int -\frac{1}{y}dy}$

$= e^{-\log y} = e^{\log\frac{1}{y}}$

$= \dfrac{1}{y}$

Sol^n is

$x\cdot\dfrac{1}{y} = \int \cancel{\dfrac{1}{y}} \times 2y^{\cancel{2}}\, dy + C$

$\dfrac{x}{y} = \cancel{2}\times\dfrac{y^2}{\cancel{2}} + C$

$$P = \frac{1}{y^2}, \quad Q = \frac{1}{y^3}$$

$$IF = e^{\int y^{-2}\,dy}$$

$$= e^{-\frac{1}{y}}$$

Solⁿ. is

$$x \times e^{-\frac{1}{y}} = \int e^{-\frac{1}{y}} \times \frac{1}{y^3}\,dy + \text{const} \qquad (1)$$

$$\int e^{-\frac{1}{y}} \times \frac{1}{y^3}\,dy$$

$$= -\int e^{t}\, t\, dt$$

Put $t = -\frac{1}{y}$

$dt = \frac{1}{y^2}\,dy$

Prob: $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$

$$\Rightarrow \left(x - \frac{1}{y}\right)\frac{dy}{dx} = -y^2$$

$$\Rightarrow \left(x - \frac{1}{y}\right) = -y^2\frac{dx}{dy}$$

$$\Rightarrow -\frac{1}{y^2}x + \frac{1}{y^3} = \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3}$$

$$x \times e^{-1/y} = \left(\frac{1}{y}+1\right)e^{-1/y} + C$$

$$x = \left(\frac{1}{y}+1\right) + C\cdot e^{1/y}$$

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prob: $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = 0$

HW

Put $z = \frac{dy}{dx}$

Ans: $\boxed{y + x = c_1 \log x + c_2}$

$$= -\int t e^t dt$$

$$= -\left[t\int e^t dt - \int 1\cdot e^t dt\right]$$

$$= -\left[te^t - e^t\right]$$

$$= -\left[-\frac{1}{y}e^{-1/y} - e^{-1/y}\right]$$

$$= \frac{1}{y}e^{-1/y} + e^{-1/y}$$

Using this in (1), we have