KGS IAS

Maths Optional

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$+\frac{1}{\lfloor 2}\Big(0+0+2a_2+6a_3x$

$+\cdots+a_n n(n-1)x^{n-1}\Big)$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$+\frac{1}{\lfloor n}\Big(0+0-- \quad +\lfloor n\, a_n\Big)$

$= a_0+a_1(x+1)$

$+a_2(x+1)^2+a_3(x+1)^3$

$\cdots +a_n(x+1)^n$

$= \underline{f(x+1)} = T(f(x))$

→ Hints to the HW (last class)

Let $f(x)\in P_n(\mathbb{R})$

$f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

$a_i\in\mathbb{R}$

$$\left(1+\frac{D}{\lfloor 1}+\frac{D^2}{\lfloor 2}+\cdots+\frac{D^n}{\lfloor n}\right)f(x)$$

$$=\left(1+\frac{D}{\lfloor 1}+\frac{D^2}{\lfloor 2}+\cdots+\frac{D^n}{\lfloor n}\right)\left(a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_nx^n\right)$$

$$=\left(a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_nx^n\right)$$

$$+\frac{1}{\lfloor 1}\left(0+a_1+2a_2x+3a_3x^2+\cdots+a_n n x^{n-1}\right)$$

$\Rightarrow T(0) + T(0) = T(0) + 0$

$\Rightarrow T(0) = 0$ $\left(\because (U, +) \text{ is an abelian group}\right)$

By Left cancellation law.

(ii) $T(-u) = T((-1) \cdot u)$

$= (-1) \cdot T(u)$

$= -T(u).$

Properties of L.T.

Let $T : U \to V$ be a L.T. Then

(i) $T(0) = 0$

(ii) $T(-u) = -T(u). \quad \forall u \in U$

(iii) $T(u - v) = T(u) - T(v)$

Proof. (i)

$0 + 0 = 0$

$\Rightarrow T(0+0) = T(0)$

$\Rightarrow T(0) + T(0) = T(0)$

be a set of n vectors in V. Then there exists a unique linear transformation $T: U \to V$ s.t.

$$T(u_i) = v_i \quad \forall i = 1, 2, \ldots n.$$

Proof: $\because S = \{u_1, u_2, \ldots u_n\}$ is a basis of U.

$\therefore L(S) = U$.

Let $u \in U$

$\therefore u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

$\alpha_i \in F$.

(iii) $T(u - v) = T(u + (-v))$

$$= T(u) + T(-v)$$

$$= T(u) - T(v)$$

Determination of a L.T.

Let U and V be two vector space over the field F and $S = \{u_1, u_2, \ldots u_n\}$ be a basis of U. Let $\{v_1, v_2, \ldots v_n\}$

Claim: T is a L.T.

Let $u, v \in U$

$a, b \in F$.

$\therefore u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$

$v = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_n u_n$

$\alpha_i, \beta_i \in F$

Now

$au + bv = a(\alpha_1 u_1 + \dots + \alpha_n u_n)$

$+ b(\beta_1 u_1 + \dots + \beta_n u_n)$

$= (a\alpha_1 + b\beta_1) u_1 + \dots$

$+ (a\alpha_n + b\beta_n) u_n$

Define $T : U \to V$ as:

$T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n)$

$= \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

$\because u_i = 0 \cdot u_1 + 0 \cdot u_2 + \dots + 1 \cdot u_i + 0 \cdot u_{i+1} + \dots + 0 \cdot u_n$.

$T(u_i) = 1 \cdot v_i = v_i$

$\forall i = 1, 2, \dots n$.

$$= aT(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n)$$

$$+ bT(\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n)$$

$$= aT(u) + bT(v)$$

T is unique:

Define a L.T

$S: U \to V$ as

$S(u_i) = v_i \quad \forall i = 1, 2, \ldots, n$

$$T(au + bv)$$

$$= T\left[(a\alpha_1 + b\beta_1)u_1 + (a\alpha_2 + b\beta_2)\cdot u_2 + \cdots (a\alpha_n + b\beta_n)u_n\right]$$

$$= (a\alpha_1 + b\beta_1)v_1 + (a\alpha_2 + b\beta_2)v_2 + \cdots + (a\alpha_n + b\beta_n)v_n.$$

$$= a\left[\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n\right]$$

$$+ b\left[\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n\right]$$

$= T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n)$

$= T(u)$

$\therefore S(u) = T(u) \quad \forall u \in U$

$\therefore S = T$

Let $u \in U$

$\therefore u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

$\alpha_i \in F.$

$S(u) = S(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n)$

$= \alpha_1 S(u_1) + \alpha_2 S(u_2) + \cdots + \alpha_n S(u_n)$

$= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ [$\because$ S is L.T]

Sum of Linear transformations

Let T_1 & T_2 be two linear transformations from $U(F)$ into $V(F)$. Then their sum $T_1 + T_2$ is defined as:

$$(T_1 + T_2)(u) = T_1(u) + T_2(u) \quad \forall\, u \in U$$

→ $T_1 + T_2$ is a L.T. (prove it)

$$(T_1 + T_2)(au + bv) = a(T_1 + T_2)(u) + b \cdot (T_1 + T_2)(v)$$

→ αT is a L.T. (prove it)

$$(\alpha T)(au+bv)$$
$$= a\left[(\alpha T)(u)\right] + b\left[(\alpha T)(v)\right]$$

$$(\alpha T)(au+bv)$$
$$= \alpha \cdot T(au+bv)$$
$$= \alpha\left[aT(u)+bT(v)\right]$$

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Scalar multiplication of a L.T.

Let $T: U(F) \to V(F)$ be a L.T. and $\alpha \in F$. Then the function αT defined as:

$$(\alpha T)(u) = \alpha \cdot T(u) \quad \forall u \in U$$

$\rightarrow$ (i) $(H+T)(x,y,z)$

$$= H(x,y,z) + T(x,y,z)$$

$$= (2x, y-z) + (x-y, y+z)$$

$$= (3x-y,\ 2y)$$

(ii) $(aH)(x,y,z)$

$$= (2ax,\ ay-az)$$

HW

Prob: Let $T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ and $H: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ be the two linear transformations defined by

$$T(x,y,z) = (x-y,\ y+z)$$

& $$H(x,y,z) = (2x,\ y-z)$$

Find (i) $H+T$

(ii) aH.

prob: $G: V_3 \to V_3$ and $H: V_3 \to V_3$ be two linear operators defined by

$G(e_1) = e_1 + e_2$, $G(e_2) = e_3$,

$G(e_3) = e_2 - e_3$

&

$H(e_1) = e_3$, $H(e_2) = 2e_2 - e_3$,

$H(e_3) = 0$

Where $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(\mathbb{R})$.

Find (i) $G + H$ (ii) $2G$

$\rightarrow$ Let $(x, y, z) \in V_3$

$\therefore (x, y, z) = xe_1 + ye_2 + ze_3$

$(G+H)(x, y, z)$

$= G(x, y, z) + H(x, y, z)$

$= G[xe_1 + ye_2 + ze_3]$

$+ H(xe_1 + ye_2 + ze_3)$

$= xG(e_1) + yG(e_2) + zG(e_3)$

$+ xH(e_1) + yH(e_2) + zH(e_3)$

(11) let $(x, y, z) \in V_3$

$\therefore (x, y, z) = x e_1 + y e_2 + z e_3$

$(2\varphi)(x, y, z)$

$= 2 \cdot \varphi(x e_1 + y e_2 + z e_3)$

$= 2 \cdot [x \varphi(e_1) + y \cdot \varphi(e_2) + z \cdot \varphi(e_3)]$

$\vdots$

$= (2x, 2x + 2z, 2y - 2z)$

$= x(e_1 + e_2) + y \cdot e_3 + z(e_2 - e_3)$

$+ x \cdot e_3 + y(2e_2 - e_3)$

$+ z \cdot 0$

$= x \cdot (e_1 + e_2 + e_3) + y \cdot (e_3 + 2e_2 - e_3)$

$+ z(e_2 - e_3)$

$= x \cdot (1, 1, 1) + y(0, 2, 0)$

$+ z(0, 1, -1)$

$= (x, x + 2y + z, x - z)$

$(TH)(v) = T(H(v))$

$\forall\, v \in U$.

is a L.T. from U into W.

Product of linear transformations:

Let $U(F)$, $V(F)$, $W(F)$ be the three vector spaces. $T: V \to W$ and $H: U \to V$ be the two linear transf. Then the composite fn. TH (called the product of L.T.'s) defined by proof:

(I)

$u \in U(F)$

$$\left[T(H+H')\right](u)$$
$$= T\left[(H+H')(u)\right]$$
$$= T\left[\underline{H(u)} + \underline{H'(u)}\right]$$
$$= T\left(H(u)\right) + T\left(H'(u)\right)$$
$$= TH(u) + TH'(u)$$
$$= (TH + TH')(u)$$

Result: Let H, H' be two L.T.'s from $U(F)$ to $V(F)$. Let T, T' be the L.T.'s from $V(F)$ to $W(F)$ and $a \in F$, then

(I) $T(H+H') = TH + TH'$

(II) $(T+T')H = TH + T'H$.

(III) $a(TH) = (aT)H = T\cdot(aH)$.

(IV) $(A+B)\cdot C = AC + BC$.

(V) $A(BC) = (AB)\cdot C$.

→ (III) <u>proof</u>

let $u \in V$

$$[A\cdot(B+C)](u)$$
$$= A[(B+C)(u)]$$
$$= A[B(u)+C(u)]$$
$$= A(B(u)) + A(C(u))$$
$$= AB(u) + AC(u)$$

Algebra of Linear operators

Let A, B, C be linear operators on a vector space V(F). Also let 'O' be the zero operator and 'I' the identity operator on V. Then

(I) $A\cdot O = O\cdot A = O$

(II) $A\cdot I = I\cdot A = A$

(III) $A\cdot(B+C) = AB + AC$

prob: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ and $H: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (3x, y+z)$ and $H(x, y, z) = (2x - z, y)$. Compute

(i) $4T - 5H$ (ii) TH (iii) HT

$\to$ (i) $(4T - 5H)(x, y, z)$

HW

$$= (2x + 5z, -y + 4z)$$

$\to$ (ii) $H: \mathbb{R}^3 \to \underline{\mathbb{R}^2}$, $T: \mathbb{R}^3 \to \mathbb{R}^2$

TH, HT not define.

$$= (AB + AC)(u)$$

$$\therefore [A \cdot (B + C)](u)$$
$$= (AB + AC)(u)$$
$$\forall u \in V.$$

$$\therefore A \cdot (B + C) = AB + AC$$

→ $T_1 \cdot T_2$ not defined

$T_2 \cdot T_1 : \mathbb{R}^3 \to \mathbb{R}^2$

$$(T_2 \cdot T_1)(x, y, z)$$

$$= T_2\left[T_1(x, y, z)\right]$$

$$= T_2\left[3x, 4y - z\right]$$

$$= (-3x, 4y - z)$$

<u>prob:</u> Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are the two linear transformations defined by

$$T_1(x, y, z) = (3x, 4y - z)$$

$$T_2(x, y) = (-x, y).$$

Compute if possible $T_1 \cdot T_2$ & $T_2 \cdot T_1$

(1) $TD \neq DT$

(11) $(TD)^2 = T^2D^2 + TD$

Prob: Let $P(\mathbb{R})$ be the vector space of all polynomials in 'x' over the field $\mathbb{R}$. D, T be two linear operators on $P(\mathbb{R})$, defined by $D(f(x)) = \frac{d}{dx}(f(x))$, & $T(f(x)) = x f(x)$, $\forall f(x) \in P(\mathbb{R})$

Show that