$r(AB)=1$

Note: If A is a non-zero column matrix and B is a non-zero row matrix then $r(AB)=1$.

$\Rightarrow$

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ | \\ a_{m1} \end{bmatrix}_{m\times 1} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & - & b_{1n} \end{bmatrix}_{1\times n}$$

$$AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & - & - & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & - & - & a_{21}b_{1n} \\ \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & - & - & a_{m1}b_{1n} \end{bmatrix}_{m\times n}$$

$$\begin{vmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{vmatrix}$$

$$= a_{11}b_{11}a_{21}b_{12} - a_{21}b_{11}a_{11}b_{12}$$

$$= 0$$

$$A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{2\times 1} \qquad B = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}_{1\times 3}$$

$$(AB)_{2\times 3} = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$$

$$\underline{r(AB) = 1}$$

Prob: If $U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$

then find the rank of U and U^2.

Sol^n: U —— Echelon matrix.

$\therefore r(U) =$ No. of non-zero rows of U.

$= 3.$

$U^2 = U \cdot U$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

Echelon form; $\therefore r(U^2) = 2$

Partitioning of a matrix

$$A = \left[\begin{array}{cc|ccc} 1 & 2 & 1 & 2 & -1 \\ 2 & 3 & 2 & 3 & 1 \\ 3 & -1 & 0 & 4 & 1 \\ \hline 4 & 0 & 1 & 0 & 1 \end{array}\right]_{4\times 5}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & -1 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 4 & 0 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

Ex: $A = \left[\begin{array}{cc|ccc} 1 & 2 & 3 & 4 & 0 \\ 5 & -2 & 0 & 1 & 1 \\ \hline 3 & 1 & 2 & -1 & 1 \end{array}\right]$ 3×5

$B = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & 0 & -1 & -2 \\ \hline 2 & -1 & 2 & 1 & 0 \end{array}\right]$ 3×5

$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

Matrices partitioned identically for addition

$A \,,\, B$ — order same.

A_{ij} B_{ij} — order same.

$A+B = [A_{ij} + B_{ij}]$

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}$$

Matrices partitioned conformably for multiplication

Ex: $A = \left[\begin{array}{cc|cc|c} 1 & -1 & 0 & 2 & 3 \\ 2 & 1 & -1 & 0 & 0 \\ \hline 0 & 1 & 2 & -2 & 3 \end{array}\right]_{3\times 5}$

$B = \left[\begin{array}{ccc|cc} 1 & -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 3 & -3 \\ \hline -1 & 1 & 0 & 0 & 2 \\ 1 & 2 & 3 & 0 & 2 \\ \hline -1 & 1 & 1 & -1 & 3 \end{array}\right]_{5\times 5}$

$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$

$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$

$A = [a_{ij}]_{m\times n}$ $\quad$ $B = [b_{ij}]_{n\times p}$.

partition arbitrarily

column || lines

Accordingly

Draw lines || to rows of B

$$AB = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}_{2\times 3} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} A_{11}\cdot B_{11} + A_{12}\cdot B_{21} + A_{13}\cdot B_{31} & \text{—} \\ \text{—} & \text{—} \end{bmatrix}_{2\times 2}$$

Prob: If P, Q are non-singular matrices then show that if $A = \begin{bmatrix} P & O \\ O & Q \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} P^{-1} & O \\ O & Q^{-1} \end{bmatrix}$

Sol^n: Let $B = \begin{bmatrix} M & R \\ N & S \end{bmatrix}$ be the inverse of A. B is partitioned in such a way that AB is possible.

$$AB = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$$\begin{bmatrix} P & O \\ O & Q \end{bmatrix} \begin{bmatrix} M & R \\ N & S \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} PM & PR \\ QN & QS \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$(IV) \Rightarrow \quad QS = I$

$Q^{-1}(QS) = Q^{-1}I$

$\Rightarrow (Q^{-1}Q)S = Q^{-1}$

$\Rightarrow I \cdot S = Q^{-1}$

$\Rightarrow \quad S = Q^{-1}.$

$(II) \Rightarrow$

$PR = 0$

$P^{-1}(PR) = P^{-1} \cdot 0$

$\Rightarrow (P^{-1}P)R = 0$

$\Rightarrow IR = 0 \Rightarrow R = 0$

$PM = I \text{ — (I)} \quad PR = 0 \text{ — (II)}$

$QN = 0 \text{ — (III)} \quad QS = I \text{ — (IV)}$

$\because$ P and Q are non-singular matrices.

$\therefore P^{-1}$ and Q^{-1} exist.

From (I)

$PM = I \Rightarrow P^{-1}(PM) = P^{-1} \cdot I$

$\Rightarrow (P^{-1}P)M = P^{-1}$

$\Rightarrow IM = P^{-1}$

$\Rightarrow M = P^{-1}$

$$A^{-1} = B = \begin{bmatrix} \varphi^{-1} & 0 \\ 0 & \varphi^{-1} \end{bmatrix}$$

III $\Rightarrow$

$$\varphi N = 0$$

$$\Rightarrow \varphi^{-1}(\varphi N) = \varphi^{-1} \cdot 0$$

$$\Rightarrow (\varphi^{-1}\varphi)N = 0$$

$$\Rightarrow IN = 0$$

$$\Rightarrow N = 0$$

$$TM = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} P & R \\ Q & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Prob: Find the inverse of $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ where B, C are non-singular.

→ Let $M = \begin{bmatrix} P & R \\ Q & S \end{bmatrix}$ be the inverse of $T = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ and M is partitioned conformably for multiplication with T.

(IV) $\Rightarrow$ $CR = I$

$$C^{-1}(CR) = C^{-1}\cdot I$$

$$\Rightarrow \underline{R = C^{-1}}.$$

(III) $\Rightarrow$ $CP = 0$

$$C^{-1}(CP) = C^{-1}\cdot 0$$

$$\Rightarrow P = 0$$

(I) $\Rightarrow$ $AP + BQ = I$

$$\underline{A\cdot 0} + BQ = I$$

$$BQ = I$$

$$B^{-1}(BQ) = B^{-1}\cdot I$$

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} P & R \\ Q & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} AP+BQ & AR+BS \\ CP & CR \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$\Rightarrow AP + BQ = I$ — (I) $\quad AR + BS = 0$ — (II)

$CP = 0$ — (III) $\quad CR = I$ — (IV)

$\because$ B and C are non-singular.

$\therefore$ B^{-1} and C^{-1} exist.

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$$

$$\Rightarrow Q = B^{-1}.$$

(ii) $AR + BS = 0$

$$\Rightarrow AC^{-1} + BS = 0$$

$$\Rightarrow BS = -AC^{-1}$$

$$\Rightarrow B^{-1}(BS) = -B^{-1}AC^{-1}$$

$$\Rightarrow (B^{-1}B)S = -B^{-1}AC^{-1}$$

$$\Rightarrow S = -B^{-1}AC^{-1}.$$

Elementary matrices

A matrix obtained from a unit matrix by a single elementary transformation is called an elementary matrix (E-matrix).

Ex (1) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad R_1 \leftrightarrow R_2$

(11) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad C_1 \to C_1 + 2C_2$

Notation:

(II) $R_i \to kR_i \; (C_i \to kC_i)$

$E_i(k)$

Ex: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E_1(2) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $R_1 \to 2R_1$

(I) $R_i \leftrightarrow R_j \; (C_i \leftrightarrow C_j)$

E_{ij}

Ex: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $C_1 \leftrightarrow C_2$

$c_i \to c_i + k c_j$

$E'_{ij}(k)$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E'_{12}(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad c_1 \to c_1 + 3c_2$$

(III) $R_i \to R_i + k R_j$

$E_{ij}(k)$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{23}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2 \to R_2 - 2R_3$$

Ex: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\rightarrow |E_{ij}| = -1$

$\rightarrow |E_i(k)| = k \quad (k \neq 0)$

$\rightarrow |E_{ij}(k)| = 1 \quad (k \neq 0)$

$\rightarrow |E'_{ij}(k)| = 1$

$$|E_{21}(3)| = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad R_2 \to R_2 + 3R_1$$

$$= 1.$$

Note: All elementary matrices are non-singular.