KGS IAS

Maths Optional

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prob: $\longrightarrow$ $2y\,dx + x\,dy = 0$

$M\,dx + N\,dy = 0$

$M = 2y \qquad N = x$

$$\frac{\partial M}{\partial y} = 2 \qquad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Not exact.

$2y\,dx + x\,dy = 0$

$y \cdot \boxed{2x\,dx} + x^2\,dy = 0$

'x' (Multiply) $\downarrow$ I.F.

$y \cdot d(x^2) + x^2\,dy = 0$

$\Rightarrow \quad d(x^2 y) = 0$

integrate,

$x^2 y = \text{const.}$

Exact.

Solⁿ.

$y\int 2x\,dx + 0 = 0$

$\Rightarrow y \cdot x^2 = c$

$I.F. = x$

2nd Method

$2y\,dx + x\,dy = 0$

Multiply by x

$2xy\,dx + x^2\,dy = 0$

$M_1\,dx + N_1\,dy = 0$

$M_1 = 2xy, \; N_1 = x^2$

$\frac{\partial M_1}{\partial y} = 2x = \frac{\partial N_1}{\partial x}$

Integrate,

$\log(xy) = -\log x + \text{const}$

$\Rightarrow \log(xy) + \log x = \text{const}.$

$\Rightarrow \log(x^2 y) = \text{const}.$

$\Rightarrow \boxed{x^2 y = c}$

3rd Method:

$2y\,dx + x\,dy = 0$

$\Rightarrow y\,dx + x\,dy = -y\,dx$

$\Rightarrow d(xy) = -y\,dx$

$\Rightarrow \dfrac{d(xy)}{xy} = -\dfrac{dx}{x}$

$$\frac{1}{x^2}(y\,dx - x\,dy) = 0$$

$$\Rightarrow \frac{y}{x^2}dx - \frac{1}{x}dy = 0$$

$$M_1 dx + N_1 dy = 0$$

$$M_1 = \frac{y}{x^2}, \quad N_1 = -\frac{1}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{x^2}, \quad \frac{\partial N_1}{\partial x} = \frac{1}{x^2}$$

$\therefore$ Exact

Prob: $y\,dx - x\,dy = 0$

$$\rightarrow \quad M\,dx + N\,dy = 0$$

$$M = y \qquad N = -x$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Not exact.

$\therefore IF = \frac{1}{x^2}$

$\left[2nd\ IF = \frac{1}{y^2}\right]$

<u>2nd Method.</u>

$y\,dx - x\,dy = 0$

$\Rightarrow \frac{y\,dx - x\,dy}{y^2} = 0$

$\Rightarrow d\left(\frac{x}{y}\right) = 0$

Int

$\frac{x}{y} = c.$

Soln. is

$y\int x^{-2}\,dx = const$

$\Rightarrow y \cdot \frac{x^{-2+1}}{-2+1} = const.$

$\Rightarrow -\frac{y}{x} = const$

$\Rightarrow \frac{y}{x} = c$

$\boxed{y = cx}$

prob: $x\,dy - y\,dx = x(x^2+y^2)\,dx$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2+y^2} = x\,dx$$

$$\Rightarrow d\left(\tan^{-1}\frac{y}{x}\right) = x\,dx$$

Int.

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{x^2}{2} + \text{const}$$

$$2\tan^{-1}\left(\frac{y}{x}\right) - x^2 = c$$

prob:

$$x\,dy - y\,dx + 2x^3\,dx = 0$$

$$\Rightarrow x\,dy - y\,dx = -2x^3\,dx$$

$$\Rightarrow \frac{-y\,dx + x\,dy}{x^2} = -2x\,dx$$

$$\Rightarrow d\left(\frac{y}{x}\right) = -2x\,dx$$

Int

$$\frac{y}{x} + x^2 = \text{const}$$

Int.

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{x^2}{2} + \text{const}$$

$$\Rightarrow 2\tan^{-1}\left(\frac{y}{x}\right) - x^2 = c.$$

2nd Method

$$x\,dy - y\,dx = x\left(x^2+y^2\right)dx$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = \frac{1}{x}\left(x^2+y^2\right)dx$$

$$\Rightarrow d\left(\frac{y}{x}\right) = \frac{x^2}{x}\left(1+\frac{y^2}{x^2}\right)dx$$

$$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = x\,dx$$

$$\Rightarrow \frac{(y+1)}{x} - x + \cos y = \text{const.}$$

$$\Rightarrow (y+1) - x^2 + x\cos y = cx$$

Prob: $y\,dx - x\,dy + (1+x^2)\,dx + x^2 \sin y\,dy = 0$

$$\Rightarrow \frac{y\,dx - x\,dy}{x^2} + \left(\frac{1}{x^2}+1\right)dx + \sin y\,dy = 0$$

$$\Rightarrow -d\left(\frac{y}{x}\right) + (x^{-2}+1)\,dx + \sin y\,dy = 0$$

Int.

$$-\frac{y}{x} - \frac{1}{x} + x - \cos y = \text{const}$$

$$3y(1+\cos^2 x) + y^3 = C.$$

<u>proof:</u> $y \sin 2x \, dx = (1+y^2+\cos^2 x)\, dy$

$$\Rightarrow \underline{y \sin 2x \, dx - \cos^2 x \, dy} = (1+y^2)\, dy$$

$$\Rightarrow d(-y\cos^2 x) = (1+y^2)\, dy$$

and.

$$-y\cos^2 x = y + \frac{y^3}{3} + \text{const.}$$

$$\Rightarrow y(\cos^2 x + 1) + \frac{y^3}{3} = \text{const.}$$

$$\sec^2\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right) = \frac{dx}{x}$$

Int.

$$\int \sec^2\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right) = \int \frac{dx}{x}$$

+ const

$$\Rightarrow \tan\left(\frac{y}{x}\right) = \log x + \text{const}$$

$$\Rightarrow \tan\left(\frac{y}{x}\right) - \log x = c$$

Prob:

$$x\,dy = \left[y + x\cos^2\left(\frac{y}{x}\right)\right] dx$$

$$\Rightarrow x\,dy - y\,dx = x\cos^2\left(\frac{y}{x}\right) dx$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = \frac{1}{x}\cdot\cos^2\left(\frac{y}{x}\right) dx$$

$$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{\cos^2\left(\frac{y}{x}\right)} = \frac{dx}{x}$$

$$\Rightarrow \frac{d(xy)}{(xy)^2} = -\frac{2}{x}dx + \frac{1}{y}dy$$

Int.

$$\int (xy)^{-2} d(xy) = -2\log x + \log y + \text{const}$$

$$\Rightarrow -\frac{1}{xy} = -\log x^2 + \log y + \text{const}$$

$$\Rightarrow -\frac{1}{xy} = \log\left(\frac{y}{x^2}\right) + \text{const.}$$

$$\Rightarrow \frac{1}{xy} = -\log\left(\frac{y}{x^2}\right) + \text{const}$$

Prob:

$$(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$$

$$\Rightarrow xy\left(y + 2xy^2\right)dx + xy\left(x - x^2y\right)dy = 0$$

$$\Rightarrow (y + 2xy^2)dx + (x - x^2y)dy = 0$$

$$\Rightarrow y\,dx + x\,dy = -2xy^2dx + x^2y\,dy$$

$$\Rightarrow d(xy) = xy\left(-2y\,dx + x\,dy\right)$$

$$\Rightarrow \frac{d(xy)}{xy} = -2y\,dx + x\,dy$$

HW

Prob: $x\,dy - (y-x)\,dx = 0$

Ans: $\frac{y}{x} + \log x = c$

$$\frac{1}{xy} = \log\left(\frac{x^2}{y}\right) + \log c$$

$$\Rightarrow \frac{1}{xy} = \log\left(c\cdot\frac{x^2}{y}\right)$$

$$\Rightarrow \boxed{c\cdot\frac{x^2}{y} = e^{\frac{1}{xy}}}$$

$$\Rightarrow \frac{d(xy)}{\sqrt{1-(xy)^2}} = \frac{dx}{x}$$

Int

$$\sin^{-1}(xy) = \log x + c.$$

Prob:

$$x^2 \frac{dy}{dx} + xy = \sqrt{1-(xy)^2}$$

$$\Rightarrow x^2 dy + xy\, dx = \sqrt{1-(xy)^2}\, dx$$

$$\Rightarrow x\left[x\,dy + y\,dx\right] = \sqrt{1-(xy)^2}\, dx$$

$$\Rightarrow \frac{x \cdot d(xy)}{\sqrt{1-(xy)^2}} = dx$$

$M = x^2 y \qquad N = -(x^3 + y^3)$

$$\frac{\partial M}{\partial y} = x^2, \quad \frac{\partial N}{\partial x} = -3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

(1) is not exact.

$$Mx + Ny = \cancel{x^3}y - \cancel{x^3}y - y^4$$
$$= -y^4 \neq 0$$

$$IF = \frac{1}{Mx + Ny} = -\frac{1}{y^4}$$

Rules for finding integrating factors:

Rule 1: If $M\,dx + N\,dy = 0$ is homogeneous and $Mx + Ny \neq 0$ then $\frac{1}{Mx + Ny}$ is an I.F.

Prob: $x^2 y\,dx - (x^3 + y^3)\,dy = 0$ — (1)

$M\,dx + N\,dy = 0$

Rule 2: If $M\,dx + N\,dy = 0$ is expressed as

$$y \cdot f_1(x, y)\,dx + x\,f_2(x, y)\,dy = 0$$

and $Mx - Ny \neq 0$ then

$\dfrac{1}{Mx - Ny}$ is I.F.

Multiplying ① with $-\dfrac{1}{y^4}$

$$-\frac{x^2}{y^3}\,dx + \frac{1}{y^4}(x^3 + y^3)\,dy = 0$$

$$\Rightarrow -\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

Soln.

$$-\frac{x^3}{3y^3} + \log y = c$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ (1) is not exact.

$$Mx - Ny = x\cancel{y} + 2x^2y^2$$
$$- [\cancel{x}y - 2x^2y^2]$$
$$= 4x^2y^2 \neq 0$$

$$I.F. = \frac{1}{Mx - Ny} = \frac{1}{4x^2y^2}$$

Prob: $y(1+2xy)dx + x(1-2xy)dy = 0$ (1)

$$\boxed{M\,dx + N\,dy = 0}$$

$$M = y + 2x \cdot y^2$$

$$\frac{\partial M}{\partial y} = 1 + 4xy$$

$$N = x - 2x^2y$$

$$\frac{\partial N}{\partial x} = 1 - 4xy$$

$$\frac{1}{2}\log x - \frac{1}{4xy} - \frac{1}{2}\log y = \text{const}$$

$$\Rightarrow \frac{1}{2}\log\frac{x}{y} - \frac{1}{4xy} = \text{const}$$

$$\Rightarrow 2\log\frac{x}{y} - \frac{1}{xy} = C$$

Multiplying ① with $\frac{1}{4x^2y^2}$

$$\frac{y(1+2xy)dx}{4x^2y^2} + \frac{x}{4x^2y^2}(1-2xy)dy = 0$$

$$\Rightarrow \left[\frac{1}{4x^2y} + \frac{2x}{4x^2}\right]dx + \frac{x}{4x^2y^2}(1-2xy)dy = 0$$

Solⁿ. is

$$\frac{1}{-4xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = \text{const}$$