Maths Optional

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(HW)

Left class

$T : V_4 \to V_3$

Range T = Span$\{(1,1,1), (1,-1,1), (1,0,0), (1,0,1)\}$

For basis, Take these vectors as the rows of a matrix.

$\to$ $T(a,b,c,d) = T(ae_1 + be_2 + ce_3 + de_4)$

$= aT(e_1) + bT(e_2) + cT(e_3) + d\cdot T(e_4)$

$= a(f_1 + f_2 + f_3) + b(f_1 - f_2 + f_3) + c\cdot f_1 + d\cdot(f_1 + f_3)$

$= a(1,1,1) + b(1,-1,1) + c(1,0,0) + d\cdot(1,0,1)$

$= (a+b+c+d,\ a-b,\ a+b+d)$

$\therefore$ Basis of $R(T)$

$$= \{(1,1,1), (0,-2,0), (0,0,-1)\}$$

$\therefore r(T) = 3$

Let $(a,b,c,d) \in N(T)$

$\therefore T(a,b,c,d) = 0$

$$(a+b+c+d,\ a-b,\ a+b+d) = (0,0,0)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - \frac{1}{2} R_2 \\ R_4 \to R_4 - \frac{1}{2} R_2 \end{matrix}$$

Rank of coeff. matrix = 3

$< $ No. of var.

So $4-3 = \underline{1}$ L.I. soln.

$-k, 0, 2k$ $\quad a = -k$

$a+b+c+d=0$

$-2b-c-d=0$

$-c=0 \Rightarrow c=0$

$2b+d=0$

Let $d=2k$

$2b=-2k$

$b=-k$

$a+b+c+d=0$

$a-b=0$

$a+b+d=0$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$\nu(T) = 1$

$$r(T) + \nu(T) = 3 + 1$$
$$= 4$$
$$= \mathrm{Dim}(V_4)$$

Verified

$$\therefore \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -k \\ -k \\ 0 \\ 2k \end{bmatrix}$$

$$= k \underbrace{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}}_{\to \text{L.I.}}$$

Basis of $N(T) = \{(-1, -1, 0, 2)\}$

$N(T) = \{k \cdot (-1, -1, 0, 2) : k \text{ is arb.}\}$

$\therefore$ Range $T = \text{Span}\{(1,2,0,-4),$
$(2,0,-1,-3), (0,0,0,0)\}$.

Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$.

$\therefore \exists$ a unique L.T. $T : \mathbb{R}^3 \to \mathbb{R}^4$

$$T(e_1) = (1,2,0,-4)$$
$$T(e_2) = (2,0,-1,-3)$$
$$T(e_3) = (0,0,0,0)$$

Prob: Find a L.T. $T : \mathbb{R}^3 \to \mathbb{R}^4$ whose range is spanned by $(1,2,0,-4)$ and $(2,0,-1,-3)$.

Solⁿ: Range T is spanned by $(1,2,0,-4)$ and $(2,0,-1,-3)$

Consider the set
$$\{(1,2,0,-4), (2,0,-1,-3), (0,0,0,0)\}$$

prob: Let V be a vector space of all 2×2 matrices over reals. Let P be a fixed matrix of V.

$$P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$ and $T : V \to V$ be a linear operator defined by $T(A) = PA$, $\forall A \in V$

Find Nullity T.

Now

$T(a, b, c)$

$= T(ae_1 + be_2 + ce_3)$

$= aT(e_1) + b\,T(e_2) + cT(e_3)$

$= a(1, 2, 0, -4) + b(2, 0, -1, -3)$

$+ c(0, 0, 0, 0)$

$= (a + 2b, 2a, -b, -4a - 3b)$

$\Rightarrow a - c = 0, \quad b - d = 0$

$-2a + 2c = 0, \quad -2b + 2d = 0$

$\Rightarrow \quad a = c, \quad b = d$

$$\therefore A = \begin{bmatrix} a & b \\ a & b \end{bmatrix} \in N(T)$$

$$\begin{bmatrix} a & b \\ a & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Sol^n:

Let $A \in N(T)$

$T(A) = 0$

$\Rightarrow PA = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a - c & b - d \\ -2a + 2c & -2b + 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

prob: Find the null space, range, rank and nullity of the L.T. $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x+y, x-y, y)$.

$\rightarrow$ Let $(x, y) \in N(T)$

$\therefore T(x, y) = (0, 0, 0)$

$\Rightarrow (x+y, x-y, y) = (0, 0, 0)$

$\therefore$ Basis of $N(T)$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Nullity $T = 2$

By Rank-Nullity th,

$$r(T) + \nu(T) = \operatorname{Dim} \mathbb{R}^2$$

$$r(T) + 0 = 2$$

$$r(T) = 2$$

$$R(T) = \{ T(x, y) : (x, y) \in \mathbb{R}^2 \}$$

$$= \{ (x+y, x-y, y) : (x, y) \in \mathbb{R}^2 \}.$$

$$x + y = 0$$

$$x - y = 0$$

$$y = 0$$

$$\Rightarrow x = 0$$

$$\therefore N(T) = \{ (0, 0) \}.$$

$$\operatorname{Dim} N(T) = 0$$

$$\nu(T) = 0$$

prob: Show that $T:\mathbb{R}^4 \to \mathbb{R}^4$ given by $T(x, y, z, t) = (2x, 3y, 0, 0)$ is a L.T. Find its rank and Nullity

$\to$ Let $a, b \in \mathbb{R}$

Let $u = (x_1, y_1, z_1, t_1) \in \mathbb{R}^4$

$v = (x_2, y_2, z_2, t_2)$

$au + bv = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2, at_1 + bt_2)$

Now $T(au + bv)$

$= T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2, at_1 + bt_2)$

$= (2(ax_1 + bx_2), 3(ay_1 + by_2), 0, 0)$

$= (2ax_1, 3ay_1, 0, 0) + (2bx_2, 3by_2, 0, 0)$

$\therefore = aT(u) + bT(v)$

$(0,0,c,d) = (0,0,c,0)$
$+ (0,0,0,d)$
$= c(0,0,1,0)$
$+ d(0,0,0,1)$

$\therefore N(T) = \text{Span}\{(0,0,1,0),$
$(0,0,0,1)\}.$

Also $(0,0,1,0)$ & $(0,0,0,1)$
are L.I. (prove it)

$\therefore$ T is a L.T.

Let $(a,b,c,d) \in N(T)$

$\therefore T(a,b,c,d) = (0,0,0,0)$

$\Rightarrow (2a, 3b, 0, 0) = (0,0,0,0)$

$\Rightarrow a = 0, \ b = 0$

$\therefore N(T) = \{(0,0,c,d) : c, d \in \mathbb{R}\}$

Prob: Find the range, rank, kernel, and Nullity of the L.T. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.

$\to$ Let $(a, b, c) \in N(T)$

$\therefore T(a, b, c) = (0, 0)$

$\Rightarrow (a+b, 2c-a) = (0, 0)$

$\therefore$ Basis of $N(T)$

$= \{(0,0,1,0), (0,0,0,1)\}$

$\therefore \text{Dim } N(T) = 2$

$\Rightarrow \nu(T) = 2$

By rank-nullity th.,

$r(t) + \nu(T) = \text{Dim } \mathbb{R}^4$

$r(t) + 2 = 4$

$r(t) = 2$

$3-2=1$ L.I. soln.

$a+b=0$

$b+2c=0$

$b=2K$

$2K+2c=0$

$\Rightarrow 2c=-2K$

$c=-K$

$a=-b=-2K$

$a+b=0$

$-a+\quad +2c=0$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2+R_1$

Rank of Coeff. mat. $=2<$ No. of var.

By Rank-Nullity th.

$$r(T)+\nu(T)=\operatorname{Dim}\mathbb{R}^3$$

$$r(T)+1=3$$

$$r(T)=2$$

$$\text{Range } T=\{(x_1+x_2,\ 2x_3-x_4):(x_1,x_2,x_3)\in\mathbb{R}^3\}$$

$$\therefore\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}-2K\\2K\\-K\end{bmatrix}$$

$$=K\begin{bmatrix}-2\\2\\-1\end{bmatrix}$$

$$N(T)=\{K(-2,2,-1):K\in\mathbb{R}\}$$

Basis of $N(T)=\{(-2,2,-1)\}$ arb.

$$\nu(T)=1$$

Proof: (I) $\Rightarrow$ (II)

Suppose $R(T) \cap N(T) = \{0\}$.

$T(T(u)) = 0$

$\Rightarrow T(u) \in N(T)$

Also $T(u) \in R(T)$

$\therefore T(u) \in R(T) \cap N(T) = \{0\}$.

$\Rightarrow T(u) = 0$

Result: Let $V(F)$ be a vector space and T be a linear operator on V. Prove that the following statements are equivalent:

(I) $R(T) \cap N(T) = \{0\}$

(II) If $T(T(u)) = 0$ then $T(u) = 0$.

Also $y \in N(T)$

$\Rightarrow T(y) = 0$

$\Rightarrow T\big(T(x)\big) = 0$

$\Rightarrow T(x) = 0$ [By our supposition]

i.e. $y = 0$

$\therefore R(T) \cap N(T) = \{0\}$.

(II) $\Rightarrow$ (I)

Suppose $T(T(v)) = 0$

$\Rightarrow T(v) = 0$

Let $y \in R(T) \cap N(T)$

$\Rightarrow y \in R(T)$ & $y \in N(T)$

$\because y \in R(T)$

$\therefore \exists x \in V$ s.t. $T(x) = y$

Singular linear trans:

A linear transformation $T : U(F) \to V(F)$ is said to be singular if the null-space of T contains at least one non-zero vector.

i.e. $N(T) \neq \{0\}$

prob: Is there a L.T. $T : \mathbb{R}^4 \to \mathbb{R}^4$ for which rank $T = 3$ and $\nu(T) = 2$.

$\longrightarrow$ impossible.

$r(T) + \nu(T) = 3 + 2 = 5 \neq \dim \mathbb{R}^4$.

Violates Rank-Nullity th.

i.e. $T(v)=0$

$\Rightarrow v=0$

In other words,

if there exists a vector

v s.t. $T(v)=0$

but $v \neq 0$

Non-singular linear transformation:

$N(T)=\{0\}$.

Result: Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \to V$ be a L.T. Then T is non-singular iff the set of images of a linearly independent set is L.I.

$\to$ Suppose T is non-singular.

Let $\{v_1, v_2, \ldots v_n\}$ be a L.I. subset of U.

To prove: $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is L.I.

Let $a_1, a_2, \ldots a_n \in F$ s.t.

$$a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n) = 0$$

$\therefore$ The set $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is L.I.

Conversely suppose that the set of images of a L.I. set is L.I.

To prove: T is non-singular.

i.e. to prove $T(u) = 0 \Rightarrow u = 0$

$\Rightarrow T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = 0$

$[\because T \text{ is L.T.}]$

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_nv_n \in N(T) = \{0\}$.

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$

$[\because v_1, v_2, \ldots, v_n$ are L.I.$]$

$\therefore T(v) \neq 0$

Thus we have proved

$$v \neq 0 \Rightarrow T(v) \neq 0$$

i.e. $T(v) = 0 \Rightarrow v = 0$

$$\therefore N(T) = \{0\}.$$

$\therefore T$ is non-singular.

Equivalently, to prove

$$v \neq 0 \Rightarrow T(v) \neq 0$$

Suppose $v \neq 0$

$\therefore \{v\}$ is L.I.

$\therefore$ By our supposition,

$\{T(v)\}$ is L.I.