KGS IAS

Maths Optional

$$\int_a^a f(x)dx = 0$$

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$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$e^{2x}\cosh x = e^{2x}\left[\frac{e^x + e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[e^{3x} + e^x\right]$$

$$D^3 - 5D^2 + 7D - 3$$

$$= (D-1)^2(D-3)$$

(HW)

Ans: $y = (C_1 + C_2 x)e^x + C_3 e^{3x}$

$$+ \frac{1}{8}\left[x e^{3x} - x^2 e^x\right].$$

(HW)

Prob Solve

$$(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$$

$m = 1, 1, 3$

$$C.F. = (C_1 + C_2 x)e^x + C_3 e^{3x}.$$

$$= \frac{1}{8}\left[x e^{3x} - x^2 e^{x}\right]$$

Prob: Solve $(D^2+a^2)y = \sec(ax)$

A.E

$$m^2 + a^2 = 0$$
$$\Rightarrow m^2 = -a^2$$
$$\Rightarrow m = \pm ai$$

$$CF = C_1 \cos ax + C_2 \sin ax$$

$$PI = \frac{1}{(D-1)^2(D-3)}\left[\frac{1}{2}(e^{3x}+e^{x})\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)^2(D-3)} e^{3x} + \frac{1}{(D-1)^2(D-3)} e^{x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} \times \frac{e^{3x}}{D-3} + \left(\frac{-1}{2}\right)\frac{1}{(D-1)^2} e^{x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} \times \frac{x}{1!} e^{3x} - \frac{1}{2} \times \frac{x^2}{1!} e^{x}\right]$$

$$\frac{1}{D-ai}\sec ax$$

$$= e^{aix}\int e^{-aix}\cdot \sec ax\,dx$$

$$= e^{ai}x\int[\cos(-ax)+i\sin(-ax)]\cdot \sec ax\,dx$$

$$= e^{aix}\int \sec ax\,[\cos ax - i\sin ax]\,dx$$

$$= e^{aix}\int\left(1-i\frac{\sin ax}{\cos ax}\right)dx$$

$$= e^{aix}\left[x+\frac{i}{a}\log\cos ax\right]$$

$$PI = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{(D-ai)(D+ai)}\sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai}-\frac{1}{D+ai}\right]\sec ax$$

$$= \frac{1}{2ai}\left[x\left(e^{iax} - e^{-iax}\right) + \frac{i}{a}\log\cos(ax)\left(e^{iax} + e^{-iax}\right)\right]$$

$$= \frac{1}{2ai}\left[x \times 2i\sin ax + \frac{i}{a}\log\cos ax\,(2\cos ax)\right]$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cdot\cos ax\cdot\log\cos ax$$

General soln.

$$y = CF + PI.$$

Similarly,

$$\frac{1}{D+ai}\sec ax$$

$$= e^{-iax}\left[x - \frac{i}{a}\log\cos ax\right]$$

$$PI = \frac{1}{2ai}\left[e^{iax}\left(x + \frac{i}{a}\log\cos ax\right) - e^{-iax}\left(x - \frac{i}{a}\log\cos ax\right)\right]$$

$D^4 \sin ax = a^4 \sin ax$

$= (-a^2)^2 \sin ax$

$(D^2)^n \sin ax = (-a^2)^n \sin ax$

$\therefore f(D^2) \sin ax$

$= f(-a^2) \sin ax$

$\Rightarrow \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$

Case (II) To find PI, when $\varphi = \sin ax$ or $\cos ax$

$f(D)\, y = \varphi$

(a) $f(-a^2) \neq 0$

$D \sin ax = a \cos ax$

$D^2 \sin ax = -a^2 \sin ax$

$D^3 \sin ax = -a^3 \cos ax$

$$2i\sin ax = e^{iax} - e^{-iax}$$

$$\Rightarrow \sin ax = \frac{1}{2i}\left[e^{iax} - e^{-iax}\right]$$

$$\frac{1}{D^2+a^2}\sin ax$$

$$= \frac{1}{(D-ai)(D+ai)} \frac{1}{2i}\left[e^{iax} - e^{-iax}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{(D-ai)(D+ai)} e^{iax} - \frac{1}{(D-ai)(D+ai)} e^{-iax}\right]$$

Similarly,

$$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$$

$$[f(-a^2) \neq 0]$$

(b) $f(-a^2) = 0$

Ex. $\frac{1}{D^2+a^2}\sin ax$

$$= -\frac{x}{4a} \times 2 \cos ax$$

$$= -\frac{x}{2a} \cos ax$$

$$\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$$

$$= \frac{1}{2i}\left[\frac{1}{2ai(D-ai)} e^{iax} + \frac{1}{2ai(D+ai)} e^{-iax}\right]$$

$$= -\frac{1}{4a}\left[\frac{e^{iax}}{D-ai} + \frac{e^{-iax}}{D+ai}\right]$$

$$= -\frac{1}{4a}\left[\frac{x}{1!} e^{iax} + \frac{x}{1!} \cdot e^{-iax}\right]$$

$$= -\frac{x}{4a}\left[e^{iax} + e^{-iax}\right]$$

A.E:

$$m^2 - m - 2 = 0$$

$$\Rightarrow m^2 - 2m + m - 2 = 0$$

$$\Rightarrow m(m-2) + 1(m-2) = 0$$

$$\Rightarrow (m+1)(m-2) = 0$$

$$m = -1, 2$$

$$C.f = c_1 e^{-x} + c_2 e^{2x}$$

Ex: $\frac{1}{D^2 + a^2} \cos ax = \frac{x \sin ax}{2a}$

(H.W)

$$e^{iax} + e^{-iax} = 2\cos ax$$

$$\Rightarrow \cos ax = \frac{1}{2}\left[e^{iax} + e^{-iax}\right]$$

Prob: Solve

$$(D^2 - D - 2)y = \sin 2x$$

$$= \frac{-(D-6)}{-2^2-36} \sin 2x$$

$$= \frac{1}{40}[D-6]\sin 2x$$

$$= \frac{1}{40}[2\cos 2x - 6\sin 2x]$$

$$= \frac{1}{20}[\cos 2x - 3\sin 2x]$$

$$\therefore \boxed{y = CF + PI}$$

$$PI = \frac{1}{D^2-D-2}\sin 2x$$

$$= \frac{1}{-2^2-D-2}\sin 2x$$

$$= \frac{-1}{D+6}\sin 2x$$

$$= \frac{-(D-6)}{D^2-36}\sin 2x$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-4}\cdot e^{0x} + \frac{1}{2}\cdot\frac{1}{D^2-4}\cos 2x$$

$$= \frac{1}{2}\cdot\frac{1}{(-4)} + \frac{1}{2}\cdot\frac{1}{-4-4}\cos 2x$$

$$= -\frac{1}{8} - \frac{1}{16}\cos 2x$$

$$= -\frac{1}{8}\left[1 + \frac{1}{2}\cos 2x\right]$$

$$y = CF + PI$$

Prob: Solve

$$(D^2-4)y = \cos^2 x = \frac{1}{2}[1+\cos 2x]$$

A.E.

$$m^2 - 4 = 0$$

$$\Rightarrow m = \pm 2$$

$$C.F. = c_1 e^{2x} + c_2 e^{-2x}$$

$$P.I = \frac{1}{D^2-4}\left[\frac{1}{2} + \frac{1}{2}\cos 2x\right]$$

$$C.F. = c_1 e^x + e^x [c_2 \cos x + c_3 \sin x]$$

Prob: Solve

$$(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$$

$$P.I = \frac{1}{D^3 - 3D^2 + 4D - 2} e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x$$

$$= \frac{1}{(D-1)(D^2 - 2D + 2)} e^x + \frac{1}{(-1^2)D - 3(-1^2) + 4D - 2} \cos x$$

$$= \frac{1}{D-1} e^x + \frac{1}{3D+1} \cos x$$

$$= x e^x + \frac{3D - 1}{9D^2 - 1} \cos x$$

A.E:

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$(m-1)(m^2 - 2m + 2) = 0$$

$$\Rightarrow m = 1, \quad m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= 1 \pm i$$

Prob: Solve

$$(D^2-4D+3)y = \sin 3x \cdot \cos 2x$$

$$= \frac{1}{2}[\sin 5x + \sin x]$$

A.E: $m^2-4m+3=0$

$\Rightarrow m^2-3m-m+3=0$

$\Rightarrow m(m-3)-1(m-3)=0$

$\Rightarrow (m-1)(m-3)=0$

$m=1, 3$

$C.F = C_1 e^x + C_2 e^{3x}$

$$= x e^x - \frac{1}{10}(3D-1)\cos x$$

$$= x e^x + \frac{1}{10}[+3\sin x + \cos x]$$

$\therefore$ General soln. is

$y = CF + PI$

$$= \frac{1}{-4}\left[\frac{2D-11}{4D^2-121}\sin 5x + \frac{2D+1}{4D^2-1}\sin x\right]$$

$$= -\frac{1}{4}\left[\frac{2D-11}{-221}\sin 5x + \frac{(2D+1)(\sin x)}{-5}\right]$$

$$= \frac{1}{4}\left[\frac{10\cos 5x - 11\sin 5x}{221} + \frac{2\cos x + \sin x}{5}\right]$$

G.S

$y = CF + PI$

$$PI = \frac{1}{D^2-4D+3} \cdot \frac{1}{2}(\sin 5x + \sin x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{D^2-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-5^2-4D+3}\sin 5x + \frac{1}{-1^2-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-2(2D+11)}\sin 5x + \frac{1}{2-4D}\sin x\right]$$

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \cdots$

$|x| < 1$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$|x| < 1$

(iii) $(1+x)^n$

$= 1 + \frac{n}{1}x + \frac{n(n-1)}{\lfloor 2}x^2$

$+ \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + \cdots$

$|x| < 1$

Case II $\quad \varphi = x^m$

m — non-negative integer.

$$PI = \frac{1}{f(D)} x^m$$

Min. powers of D

Common

$$= \frac{1}{\text{Common}\left[1 \pm \phi(D)\right]} x^m$$

$$= \frac{1}{\text{Common}} \underbrace{\left[1 \pm \phi(D)\right]^{-1}} x^m$$

$$PI = \frac{1}{D^4 - a^4}(x^4)$$

$$= \frac{-1}{a^4\left[1 - \frac{D^4}{a^4}\right]} x^4$$

$$= \frac{-1}{a^4}\left[1 - \frac{D^4}{a^4}\right]^{-1} x^4$$

$$= \frac{-1}{a^4}\left[1 + \frac{D^4}{a^4}\right] x^4$$

$$= \frac{-1}{a^4}\left[x^4 + \frac{24}{a^4}\right]$$

Prob: Solve $(D^4 - a^4)y = x^4$

A.E: $m^4 - a^4 = 0$

$\Rightarrow (m^2 - a^2)(m^2 + a^2) = 0$

$m = a, -a \qquad m = \pm ai$

$C.F. = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$

$m = 0, -1, -1$

$$C.F. = c_1 + (c_2 + c_3 x)e^{-x}$$

$$P.I. = \frac{1}{D^3+2D^2+D} e^{2x} + \frac{1}{D^3+2D^2+1}(x^2+x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D(1+2D+D^2)}(x^2+x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[1+2D+D^2\right]^{-1}(x^2+x)$$

$\therefore$ Gen soln is

$$y = CF + PI$$

Prob: Solve

$$(D^3+2D^2+D)y = e^{2x}+x^2+x$$

$\rightarrow$ A.E.

$$m^3+2m^2+m = 0$$

$$\Rightarrow m(m^2+2m+1) = 0$$

$$\Rightarrow m(m+1)^2 = 0$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[x^2 - 3x + 4\right]$$

$$= \frac{e^{2x}}{18} + \int (x^2 - 3x + 4)\,dx$$

$$= \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3}{2}x^2 + 4x$$

$\therefore$ G.S. is

$$y = CF + PI.$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[1 - (2D + D^2) + (2D + D^2)^2\right](x^2 + x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[1 - 2D - D^2 + 4D^2\right](x^2 + x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[1 - 2D + 3D^2\right](x^2 + x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[x^2 + x - 2(2x + 1) + 3 \cdot (2)\right]$$

(HW) Prob: Solve

$$(D^2 - 2D + 3)y = x^2 + \cos x$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$