Maths Optional

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$= x f'(x) + f(x)$

So, $(TD)(f(x)) \neq (DT)(f(x))$

$\therefore TD \neq DT.$

(ii) $(TD)^2 = T^2 D^2 + TD$

$(TD)^2 (f(x))$

$\rightarrow D(f(x)) = \frac{d}{dx}(f(x))$

$\rightarrow T(f(x)) = x \cdot f(x) \quad \forall f(x) \in P(\mathbb{R})$

(i) $TD \neq DT$

$(TD)(f(x)) = T(D(f(x)))$

$= T(f'(x))$

$= x \cdot f'(x)$

$(DT)(f(x)) = D(T(f(x)))$

$= D(x \cdot f(x))$

$$(T^2D^2)(f(x))$$
$$= T^2\left[D(D(f(x))\right]$$
$$= T^2\left[D f'(x)\right]$$
$$= T^2\left[f''(x)\right]$$
$$= T\left[T(f''(x))\right]$$
$$= T\left[x \cdot f''(x)\right]$$
$$= x^2 \cdot f''(x)$$

$$TD(TD(f(x)))$$
$$= TD(x \cdot f'(x))$$
$$= T(D(x \cdot f'(x)))$$
$$= T(f'(x) + x f''(x))$$
$$= \underline{x \cdot f'(x) + x^2 f''(x)}$$

Result: Let $L(U, V)$ be the set of all linear transformations from a vector space $U(F)$ into a vector space $V(F)$. Then $L(U, V)$ be a vector space relative to the operations of vector addition & scalar multiplication defined as:

$$(T^2D^2 + TD)(f(x))$$

$$= (T^2D^2)(f(x)) + (TD)(f(x))$$

$$= x^2 f''(x) + x f'(x)$$

$$\therefore (TD)^2(f(x)) = (T^2D^2 + TD)(f(x))$$

$$\forall f(x) \in P(\mathbb{R})$$

$$\therefore (TD)^2 = T^2D^2 + TD$$

Result: $L(V,W)$ be the vector space of all L.T.'s from $V(F)$ into $W(F)$.

If $\dim V = m$ and $\dim W = n$ then $\dim(L(V,W)) = mn$.

(i) $(T+H)(v) = T(v) + H(v)$

(ii) $(\alpha \cdot T)(v) = \alpha \cdot T(v)$

$\forall \alpha \in F$

$\forall v \in U$.

Note: The $L(U,V)$ is also denoted by $Hom(U,V)$.

Ex: Find the dimension of $L(\mathbb{C}^3, \mathbb{R}^2)$.

$\rightarrow$ $\mathbb{C}^3$ is a vectorspace over the field $\mathbb{C}$.

$\mathbb{R}^2$ is a vector space over the field $\mathbb{R}$.

So, $\text{Dim}\left(L(\mathbb{C}^3, \mathbb{R}^2)\right)$ is not defined.

Ex: Find the dimension of $L(\mathbb{R}^3, \mathbb{R}^2)$.

$\rightarrow$ $\text{Dim}\, \mathbb{R}^3 = 3$

$\text{Dim}\, \mathbb{R}^2 = 2$

$\therefore \text{Dim}\left(L(\mathbb{R}^3, \mathbb{R}^2)\right)$

$= 3 \times 2 = 6.$

Note: Range $T \subseteq V$.

Result: Range T is a subspace of V.

$T(0)=0 \therefore 0 \in$ Range T, Range $T \neq \phi$

Proof: Let $\alpha, \beta \in F$

$v_1, v_2 \in$ Range T.

$\therefore \exists\, u_1, u_2 \in U$

s.t $T(u_1) = v_1$

$T(u_2) = v_2$

Range and Null space of a L.T.

Let $U(F)$ and $V(F)$ be the two vector spaces. And let $T: U \to V$ be a linear transformation. The range of T is defined as the set:

$$\text{Range } T = \{T(u) : u \in U\}$$

$R(T)$

Null space or kernel: Let $U(F)$ and $V(F)$ be the vector spaces and $T: U \to V$ be a L.T. Null space of T is defined to be the set:

$$N(T) = \{u \in U : T(u) = 0\}$$

$N(T)$ —— Null space of T.

Note: $N(T) \subseteq U$.

$\alpha v_1 + \beta v_2$

$= \alpha T(u_1) + \beta T(u_2)$

$= T(\alpha u_1 + \beta u_2)$

$\in$ Range T $[\because T$ is a L.T.$]$

$\left[\begin{array}{l} u_1, u_2 \in U \\ \therefore \alpha u_1 + \beta u_2 \in U \\ U \text{ is a vector space.} \end{array} \right]$

$\therefore$ Range T is a subspace of V.

$\therefore \alpha u + \beta v \in N(T)$

So $N(T)$ is a sub-space of U.

Result: $N(T)$ is a subspace of U.

$\boxed{T(0) = 0 \therefore 0 \in N(T),\ N(T) \neq \phi}$

Proof: Let $\alpha, \beta \in F$

Let $u, v \in N(T)$

$\therefore T(u) = 0,\ T(v) = 0$

Now $T(\alpha u + \beta v)$

$= \alpha \cdot T(u) + \beta \cdot T(v)$ T is L.T.

$= \alpha \cdot 0 + \beta \cdot 0$

$= 0$

Let $S_1 = \{T(u_1), T(u_2), \dots T(u_n)\} \subseteq R(T)$

Claim: $L(S_1) = R(T)$

$\because S_1 \subseteq R(T)$

$\therefore L(S_1) \subseteq R(T)$

Let $v \in R(T)$

$\therefore \exists\, u \in U$ s.t.

$T(u) = v$

Result: Let $T: U(F) \to V(F)$ be a linear trans. If U is FDVS then the range space $R(T)$ is also a FDVS.

Proof: $\because U$ is a FDVS.

$\therefore$ It must have a finite basis.

Let $S = \{u_1, u_2, \dots u_n\}$ be a basis of U.

$\therefore v = T(u) \in L(S_1)$

$\therefore R(T) \subseteq L(S_1)$

Thus $\underline{R(T) = L(S_1)}$

$\therefore R(T)$ is a $\underline{\text{FDVS}}$.

$\because u \in U$ and S is a basis of U.

$\therefore \exists\, \alpha_1, \alpha_2, \text{—} \alpha_n \in F$

s.t.

$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

$$\therefore T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n)$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n)$$

Lin. comb. of element of S_1

[$\because T$ is L.T.]

Nullity: The nullity of T, denoted as $\nu(T)$, is $\dim N(T)$.

i.e. $\nu(T) = \dim N(T)$

Dimension of $R(T)$ and $N(T)$

Let $T : U(F) \to V(F)$ be a L.T., where U is a FDVS.

Rank: Rank of T, denoted as $\rho(T)$ or $r(T)$ is $\dim R(T)$

i.e. $r(T) = \dim R(T)$

proof: Let $\dim U = n$,
and $\dim N(T) = m$.

Let $S = \{v_1, v_2, \ldots v_m\}$ be a basis of $N(T)$

$\therefore$ S is a L.I subset of $N(T)$

$\because N(T) \subseteq U$.

$\therefore$ S is a L.I. subset of U.

$\therefore$ S can be extended to form the basis of U.

Sylvester's law
(Rank-Nullity th.)

Let U and V be two vector spaces over the field F.
and $T: U \to V$ be a L.T.
Let U be a FDVS then

$$\text{Rank}(T) + \text{Nullity}(T) = \dim U.$$

i.e. $r(T) + \nu(T) = \dim U$

$\Rightarrow T(a_{m+1}v_{m+1} + \cdots + a_n v_n) = 0$

ASTULT.

$\Rightarrow a_{m+1}v_{m+1} + a_{m+2}v_{m+2} + \cdots + a_n v_n \in N(T)$

$\because$ S is a basis of $N(T)$

$\therefore \exists\, b_1, b_2, \cdots b_m \in F$

s.t.

$a_{m+1}v_{m+1} + a_{m+2}v_{m+2} + \cdots + a_n v_n$

$= b_1 v_1 + b_2 v_2 + \cdots + b_m v_m$

Let $S_1 = \{v_1, v_2, \cdots v_m, v_{m+1}, \cdots v_n\}$ be the basis of U.

Let $S_2 = \{T(v_{m+1}), T(v_{m+2}), \cdots T(v_n)\}$

Claim: S_2 is a basis of $R(T)$.

Let $a_{m+1}T(v_{m+1}) + a_{m+2}T(v_{m+2}) + \cdots + a_n T(v_n) = 0$ —(1)

$a_i \in F$

$S_2 \subseteq R(T)$

$\therefore L(S_2) \subseteq R(T)$

Let $v \in R(T)$

$\therefore \exists\, u \in U$ s.t.

$T(u) = v$.

As $u \in U$ and S_1 is a basis of U.

$\therefore \exists\, c_1, c_2, \ldots, c_n \in F$

$\Rightarrow b_1 v_1 + b_2 v_2 + \cdots + b_m v_m - a_{m+1} v_{m+1} - a_{m+2} v_{m+2} - \cdots - a_n v_n = 0$ (1)

$\because S_1$ is a basis of U.

$\therefore$ (1) $\Rightarrow$

$b_1 = b_2 = \cdots = b_m = a_{m+1} = \cdots = a_n = 0$

i.e. $a_{m+1} = a_{m+2} = \cdots = a_n = 0$

$\therefore$ From (1), S_2 is L.I.

$\therefore v = T(u) =$ linear comb. of element of S_2

$\therefore \; v \in L(S_2)$

$\therefore \; R(T) \subseteq L(S_2)$

So $L(S_2) = R(T)$

$\therefore \; S_2$ is a basis of $R(T)$

$\text{Dim } R(T) = n - m$
$= \text{Dim } U - \text{Dim } N(T)$

$\boxed{r(T) + \nu(T) = \text{Dim } U}$

s.t.

$u = c_1 u_1 + c_2 u_2 + \cdots + c_m u_m +$
$c_{m+1} u_{m+1} + \cdots + c_n u_n$.

$\Rightarrow T(u) = T(c_1 u_1 + \cdots + c_m u_m +$
$c_{m+1} u_{m+1} + \cdots + c_n u_n)$

$= c_1 T(u_1) + \cdots + c_m T(u_m)$
$+ c_{m+1} T(u_{m+1}) + \cdots + c_n T(u_n)$
$= c_{m+1} T(u_{m+1}) + \cdots + c_n T(u_n)$ $\left[\because T \text{ is L.T.}\right]$

As $T(u_i) = 0 \; \forall i = 1, 2, \ldots m$.
S is a basis of $N(T)$.

$$= a(1,1,1) + b(-1,0,1) + c(1,2,3) + d(1,-1,-3)$$

$$R(T) = \text{Span}\{(1,1,1), (-1,0,1), (1,2,3), (1,-1,-3)\}$$

Taking these vectors as the rows of a matrix.

Prob: If $T: \mathbb{R}^4 \to \mathbb{R}^3$ is a L.T. defined by $T(a,b,c,d) = (a-b+c+d,\ a+2c-d,\ a+b+3c-3d)$ for $a,b,c,d \in \mathbb{R}$ then verify

$$\rho(T) + \nu(T) = \operatorname{Dim} \mathbb{R}^4.$$

$\to$ $T(a,b,c,d)$

$$= (a-b+c+d,\ a+2c-d,\ a+b+3c-3d)$$

$$= (a,a,a) + (-b,0,b) + (c,2c,3c) + (d,-d,-3d)$$

$\therefore$ Basis of $R(T) = \{(1,1,1), (0,1,2)\}$

$\therefore$ $\text{Dim } R(T) = 2$

Let $(a,b,c,d) \in N(T)$

$\Rightarrow T(a,b,c,d) = 0$

$\Rightarrow (a-b+c+d,\ a+2c-d,\ a+b+3c-3d) = (0,0,0)$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 + 2R_2 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

Rank of coeff. matrix $= 2 <$ No. of Var.

No. of L.I. soln $= 4 - 2 = 2$

$$a - b + c + d = 0$$

$$b + c - 2d = 0$$

Let $\left.\begin{matrix} b = k_1 \\ c = k_2 \end{matrix}\right\}$

$$a - b + c + d = 0$$

$$a \quad + 2c - d = 0$$

$$a + b + 3c - 3d = 0$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{2}k_1 - \frac{3}{2}k_2 \\ k_1 \\ k_2 \\ \frac{1}{2}k_1 + \frac{1}{2}k_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}k_1 \\ k_1 \\ 0 \\ \frac{1}{2}k_1 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2}k_2 \\ 0 \\ k_2 \\ \frac{1}{2}k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ \frac{1}{2} \end{bmatrix} + k_2 \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Basis of $N(T) = \{(\frac{1}{2}, 1, 0, \frac{1}{2}), (-\frac{3}{2}, 0, 1, \frac{1}{2})\}$.

$$k_1 + k_2 = 2d$$

$$\Rightarrow d = \frac{1}{2}k_1 + \frac{1}{2}k_2$$

$$a - k_1 + k_2 + \frac{1}{2}k_1 + \frac{1}{2}k_2 = 0$$

$$\Rightarrow a - \frac{1}{2}k_1 + \frac{3}{2}k_2 = 0$$

$$\Rightarrow a = \frac{1}{2}k_1 - \frac{3}{2}k_2.$$

prob: verify Rank-Nullity th. for the Linear map $T: V_4 \to V_3$ defined by $T(e_1) = f_1 + f_2 + f_3$, $T(e_2) = f_1 - f_2 + f_3$, $T(e_3) = f_1$, $T(e_4) = f_1 + f_3$ where $\{e_1, e_2, e_3, e_4\}$ and $\{f_1, f_2, f_3\}$ be the standard bases for V_4 & V_3 resp.

$\therefore \operatorname{Dim} N(T) = 2$

$$\rho(T) + \nu(T) = 2 + 2$$
$$= 4 = \operatorname{Dim} \mathbb{R}^4$$

verified